

Lévy Processes and Infinitely Divisible Measures in the Dual of a Nuclear Space.

C. A. Fonseca-Mora

Escuela de Matemática, Universidad de Costa Rica, San José, 11501-2060, Costa Rica.
E-mail: christianandres.fonseca@ucr.ac.cr

Abstract

Let Φ be a nuclear space and let Φ'_β denote its strong dual. In this work we establish the one-to-one correspondence between infinitely divisible measures on Φ'_β and Lévy processes taking values in Φ'_β . Moreover, we prove the Lévy-Itô decomposition, the Lévy-Khintchine formula and the existence of càdlàg versions for Φ'_β -valued Lévy processes. A characterization for Lévy measures on Φ'_β is also established. Finally, we prove the Lévy-Khintchine formula for infinitely divisible measures on Φ'_β .

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1 Introduction

This work is concerned with the study of Lévy processes and infinitely divisible measures on the dual of a nuclear space.

A Lévy process is essentially a stochastic processes with independent and stationary increments. In the case of the dual of a nuclear space, the study of some specific classes of Lévy processes, in particular of Wiener processes and stochastic analysis defined with respect to these processes received considerable attention during the decades of 1980s and 1990s (see e.g. [5, 15, 18]). However, to the extent of our knowledge the only previous work on the study of the properties of general additive (and hence Lévy) processes in the dual of some classes of nuclear spaces was carried out by Üstünel in [33]. Also, cone-additive processes in the dual of some particular Fréchet nuclear spaces were studied in [22].

On the other hand, an infinitely divisible measure is a probability measure which has a convolution n th root for every natural n . Properties of infinitely divisible measures defined on locally convex spaces were explored by several authors during the decades of 1960s and 1970s (see e.g. [8, 9, 10, 31]). Nevertheless, the author of this article is not aware of any work that studies the correspondence between Lévy processes and infinitely divisible measures in the dual of a general nuclear space.

It is for the above reasons that the aim of this paper is to gain some deeper understanding on the properties of Lévy processes that takes values in the strong dual Φ'_β of a general nuclear space Φ , and their relationship with the infinitely divisible measures defined on Φ'_β . Our main motivation is to begin with a systematic study of Lévy processes on the dual of a nuclear space which could lead to the introduction of stochastic integrals and SPDEs driven by Lévy noise in Φ'_β . Some work into this direction was carried out by the author in [11].

We start in Section 2 with some preliminary results on nuclear spaces, cylindrical and stochastic processes and Radon measures on the dual of a nuclear space. Then, in Section 3.1 we utilise

some results of Siebert [29, 30] to study the problem of embedding a given infinite divisible measure μ into a continuous convolution semigroup of probability measures Φ'_β . Later, in Section 3.2 by using recent results in [12] that provides conditions for a cylindrical process to have a càdlàg version (known as regularization theorems), we provide conditions for the existence of a càdlàg Lévy version to a given cylindrical Lévy process in Φ' or to a Φ'_β -valued Lévy process. In particular we show that if the space Φ is nuclear and barrelled, then every Lévy process in Φ'_β has a càdlàg version that is also a Lévy process.

In Section 3.3 we proceed to prove the one-to-one correspondence between Lévy processes and infinitely divisible measures on Φ'_β . Here it is important to remark that the standard argument to prove the correspondence that works on finite dimensions (see e.g. Chapter 2 in [26]) does not work in our context as the Kolmogorov extension theorem is not applicable on the dual of a general nuclear space. To overcome this situation we use a projective system version of the Kolmogorov extension theorem (see [24], Theorem 1.3.4) to show a general theorem that guarantee the existence of a cylindrical Lévy process L whose cylindrical distributions extends for each time t to the measure μ_t of the continuous convolution semigroup $\{\mu_t\}_{t \geq 0}$ in which the given infinitely divisible measure μ can be embedded. Then, for this cylindrical process L we use the results in Section 3.2 to show the existence of a Φ'_β -valued càdlàg Lévy process \tilde{L} that is a version of L , and hence the probability distribution of \tilde{L}_1 coincides with μ and then we have the correspondence. In Section 3.4 we review some properties of Wiener processes in Φ'_β .

After study in Sections 4.1 and 4.2 the basic properties of Poisson integrals defined by Poisson random measures on Φ'_β , in Sections 4.3 and 4.4 we investigate the properties of the Lévy measures on Φ'_β . In particular, we will show that Lévy measures on Φ'_β are characterized by an square integrability property expressed in terms of the norm ρ' of a Hilbert space continuously embedded in the dual space Φ'_β . Moreover, our characterization generalizes, in the context of the dual of a nuclear space, the characterization for the Lévy measure of an infinitely divisible measures obtained by Dettweiler in [8] for the case of complete Badrikian spaces.

Later, we proceed to prove in Section 4.5 the so-called Lévy-Itô decomposition for the paths of a Φ'_β -valued Lévy process. More specifically, we show that a Φ'_β -valued Lévy process $L = \{L_t\}_{t \geq 0}$ has a decomposition of the form (see Theorem 4.23):

$$L_t = t\mathbf{m} + W_t + \int_{B_{\rho'}(1)} f \tilde{N}(t, df) + \int_{B_{\rho'}(1)^c} f N(t, df), \quad \forall t \geq 0,$$

where $\mathbf{m} \in \Phi'_\beta$, ρ' is the norm associated to the square integrability property of the Lévy measure ν of L and $B_{\rho'}(1)$ is the unit ball of ρ' , $\{W_t\}_{t \geq 0}$ is a Wiener process taking values in a Hilbert space continuously embedded in the dual space Φ'_β , $\left\{ \int_{B_{\rho'}(1)} f \tilde{N}(t, df) : t \geq 0 \right\}$ is a mean-zero, square integrable, càdlàg Lévy process taking values in a Hilbert space continuously embedded in the dual space Φ'_β (small jumps part) and $\left\{ \int_{B_{\rho'}(1)^c} f N(t, df) : t \geq 0 \right\}$ is a Φ'_β -valued càdlàg Lévy process defined by means of a Poisson integral with respect to the Poisson random measure N of the Lévy process L (large jumps part).

Our Lévy-Itô decomposition improves the decomposition proved by Üstünel in [33] in two directions. First, for our decomposition we only assume that the space Φ is nuclear and we do not assume any property on the dual space W , this in contrast to the decomposition in [33] where Φ is assumed to be separable, complete and nuclear, and Φ'_β is assumed to be Suslin and nuclear. Second, we have obtained a much simpler and detailed characterization of the components of the decomposition than in [33]. In particular, contrary to the decomposition in [33] we have been able to show the independence of all the random components in our decomposition. This makes our decomposition more suitable to for example introduce stochastic integrals with respect to Lévy processes. As a consequence of our proof of the Lévy-Itô decomposition, we prove a Lévy-Khintchine formula for the characteristic function of a Φ'_β -valued Lévy process (see Theorem 4.24).

Finally, by using the one-to-one correspondence between Lévy processes and infinitely divisible measures, in Section 5 we prove the Lévy-Khintchine formula for the characteristic function of an infinitely divisible measure on Φ'_β (see Theorem 5.1). More specifically, we prove that the

characteristic function $\widehat{\mu}$ of an infinitely divisible measure μ on Φ'_β is of the form:

$$\widehat{\mu}(\phi) = \exp \left[i\mathfrak{m}[\phi] - \frac{1}{2}Q(\phi)^2 + \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1 - if[\phi] \mathbb{1}_{B_{\rho'}(1)}(f) \right) \nu(df) \right], \quad \forall \phi \in \Phi,$$

where $\mathfrak{m} \in \Phi'_\beta$, Q is a continuous Hilbertian semi-norm on Φ , and ν is a Lévy measure on Φ'_β with corresponding Hilbertian norm ρ' . Here it is important to remark that our Lévy-Khintchine formula works in a case that is not covered by the formula proved by Dettweiler in [8] because our dual space is not assumed to be a complete Badrikian space as in [8].

2 Preliminaries

2.1 Nuclear Spaces And Its Strong Dual

In this section we introduce our notation and review some of the key concepts on nuclear spaces and its dual space that we will need throughout this paper. For more information see [27, 32].

Let Φ be a locally convex space (over \mathbb{R} or \mathbb{C}). If each bounded and closed subset of Φ is complete, then Φ is said to be *quasi-complete*. On the other hand, the space Φ called a *barrelled* space if every convex, balanced, absorbing and closed subset of Φ (i.e. a barrel) is a neighborhood of zero.

If p is a continuous semi-norm on Φ and $r > 0$, the closed ball of radius r of p given by $B_p(r) = \{\phi \in \Phi : p(\phi) \leq r\}$ is a closed, convex, balanced neighborhood of zero in Φ . A continuous semi-norm (respectively a norm) p on Φ is called *Hilbertian* if $p(\phi)^2 = Q(\phi, \phi)$, for all $\phi \in \Phi$, where Q is a symmetric, non-negative bilinear form (respectively inner product) on $\Phi \times \Phi$. Let Φ_p be the Hilbert space that corresponds to the completion of the pre-Hilbert space $(\Phi/\ker(p), \tilde{p})$, where $\tilde{p}(\phi + \ker(p)) = p(\phi)$ for each $\phi \in \Phi$. The quotient map $\Phi \rightarrow \Phi/\ker(p)$ has an unique continuous linear extension $i_p : \Phi \rightarrow \Phi_p$.

Let q be another continuous Hilbertian semi-norm on Φ for which $p \leq q$. In this case, $\ker(q) \subseteq \ker(p)$. Moreover, the inclusion map from $\Phi/\ker(q)$ into $\Phi/\ker(p)$ is linear and continuous, and therefore it has a unique continuous extension $i_{p,q} : \Phi_q \rightarrow \Phi_p$. Furthermore, we have the following relation: $i_p = i_{p,q} \circ i_q$.

We denote by Φ' the topological dual of Φ and by $f[\phi]$ the canonical pairing of elements $f \in \Phi'$, $\phi \in \Phi$. We denote by Φ'_β the dual space Φ' equipped with its *strong topology* β , i.e. β is the topology on Φ' generated by the family of semi-norms $\{\eta_B\}$, where for each $B \subseteq \Phi'$ bounded we have $\eta_B(f) = \sup\{|f[\phi]| : \phi \in B\}$ for all $f \in \Phi'$. If p is a continuous Hilbertian semi-norm on Φ , then we denote by Φ'_p the Hilbert space dual to Φ_p . The dual norm p' on Φ'_p is given by $p'(f) = \sup\{|f[\phi]| : \phi \in B_p(1)\}$ for all $f \in \Phi'_p$. Moreover, the dual operator i'_p corresponds to the canonical inclusion from Φ'_p into Φ'_β and it is linear and continuous.

Let p and q be continuous Hilbertian semi-norms on Φ such that $p \leq q$. The space of continuous linear operators (respectively Hilbert-Schmidt operators) from Φ_q into Φ_p is denoted by $\mathcal{L}(\Phi_q, \Phi_p)$ (respectively $\mathcal{L}_2(\Phi_q, \Phi_p)$) and the operator norm (respectively Hilbert-Schmidt norm) is denote by $\|\cdot\|_{\mathcal{L}(\Phi_q, \Phi_p)}$ (respectively $\|\cdot\|_{\mathcal{L}_2(\Phi_q, \Phi_p)}$). We employ an analogous notation for operators between the dual spaces Φ'_p and Φ'_q .

Among the many equivalent definitions of a nuclear space (see [23, 32]), the following is the most useful for our purposes.

Definition 2.1. A (Hausdorff) locally convex space (Φ, \mathcal{T}) is called *nuclear* if its topology \mathcal{T} is generated by a family Π of Hilbertian semi-norms such that for each $p \in \Pi$ there exists $q \in \Pi$, satisfying $p \leq q$ and the canonical inclusion $i_{p,q} : \Phi_q \rightarrow \Phi_p$ is Hilbert-Schmidt.

Some examples of nuclear spaces are the following (see [32], Chapter 51 and [23], Chapter 6): $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$, $\mathcal{C}_c^\infty(K)$ (K : compact subset of \mathbb{R}^d); $\mathcal{C}_c^\infty(X)$, $\mathcal{E}(X) := \mathcal{C}^\infty(X)$, $\mathcal{E}'(X)$, $\mathcal{D}(X)$, $\mathcal{D}'(X)$ (X : open subset of \mathbb{R}^d).

Let Φ be a nuclear space. If p is a continuous Hilbertian semi-norm on Φ , then the Hilbert space Φ_p is separable (see [23], Proposition 4.4.9 and Theorem 4.4.10, p.82). Now, let $\{p_n\}_{n \in \mathbb{N}}$

be an increasing sequence of continuous Hilbertian semi-norms on (Φ, \mathcal{T}) . We denote by θ the locally convex topology on Φ generated by the family $\{p_n\}_{n \in \mathbb{N}}$. The topology θ is weaker than \mathcal{T} . We will call θ a weaker countably Hilbertian topology on Φ and we denote by Φ_θ the space (Φ, θ) . The space Φ_θ is a separable pseudo-metrizable (not necessarily Hausdorff) locally convex space and its dual space satisfies $\Phi'_\theta = \bigcup_{n \in \mathbb{N}} \Phi'_{p_n}$ (see [12], Proposition 2.4). We denote the completion of Φ_θ by $\widehat{\Phi}_\theta$ and its strong dual by $(\widehat{\Phi}_\theta)'_\beta$.

2.2 Cylindrical and Stochastic Processes

Unless otherwise specified, in this section Φ will always denote a nuclear space over \mathbb{R} .

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of equivalence classes of real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We always consider the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the topology of convergence in probability and in this case it is a complete, metrizable, topological vector space.

For two Borel measures μ and ν on Φ'_β , we denote by $\mu * \nu$ their *convolution*. Recall that $\mu * \nu(A) = \int_{\Phi' \times \Phi'} \mathbb{1}_A(x+y) \mu(dx) \nu(dy)$, for any $A \in \mathcal{B}(\Phi'_\beta)$. Denote $\nu^{*n} = \nu * \dots * \nu$ (n -times) and we use the convention $\nu^0 = \delta_0$, where δ_f denotes the Dirac measure on Φ'_β for $f \in \Phi'$.

A Borel measure μ on Φ'_β is called a *Radon measure* if for every $\Gamma \in \mathcal{B}(\Phi'_\beta)$ and $\epsilon > 0$, there exist a compact set $K_\epsilon \subseteq \Gamma$ such that $\mu(\Gamma \setminus K_\epsilon) < \epsilon$. In general not every Borel measure on Φ is Radon. We denote by $\mathfrak{M}_R^b(\Phi'_\beta)$ and by $\mathfrak{M}_R^1(\Phi'_\beta)$ the spaces of all bounded Radon measures and of all Radon probability measures on Φ'_β . A subset $M \subseteq \mathfrak{M}_R^b(\Phi'_\beta)$ is called *uniformly tight* if (i) $\sup\{\mu(\Phi'_\beta) : \mu \in M\} < \infty$, and (ii) for every $\epsilon > 0$ there exist a compact $K \subseteq \Phi'_\beta$ such that $\mu(K^c) < \epsilon$ for all $\mu \in M$. Also, a subset $M \subseteq \mathfrak{M}_R^b(\Phi'_\beta)$ is called *shift tight* if for every $\mu \in M$ there exists $f_\mu \in \Phi'_\beta$ such that $\{\mu * \delta_{f_\mu} : \mu \in M\}$ is uniformly tight.

For any $n \in \mathbb{N}$ and any $\phi_1, \dots, \phi_n \in \Phi$, we define a linear map $\pi_{\phi_1, \dots, \phi_n} : \Phi' \rightarrow \mathbb{R}^n$ by

$$\pi_{\phi_1, \dots, \phi_n}(f) = (f[\phi_1], \dots, f[\phi_n]), \quad \forall f \in \Phi'. \quad (2.1)$$

The map $\pi_{\phi_1, \dots, \phi_n}$ is clearly linear and continuous. Let M be a subset of Φ . A subset of Φ' of the form

$$\mathcal{Z}(\phi_1, \dots, \phi_n; A) = \{f \in \Phi' : (f[\phi_1], \dots, f[\phi_n]) \in A\} = \pi_{\phi_1, \dots, \phi_n}^{-1}(A) \quad (2.2)$$

where $n \in \mathbb{N}$, $\phi_1, \dots, \phi_n \in M$ and $A \in \mathcal{B}(\mathbb{R}^n)$ is called a *cylindrical set* based on M . The set of all the cylindrical sets based on M is denoted by $\mathcal{Z}(\Phi', M)$. It is an algebra but if M is a finite set then it is a σ -algebra. The σ -algebra generated by $\mathcal{Z}(\Phi', M)$ is denoted by $\mathcal{C}(\Phi', M)$ and it is called the *cylindrical σ -algebra* with respect to (Φ', M) . If $M = \Phi$, we write $\mathcal{Z}(\Phi') = \mathcal{Z}(\Phi', \Phi)$ and $\mathcal{C}(\Phi') = \mathcal{C}(\Phi', \Phi)$. One can easily see from (2.2) that $\mathcal{Z}(\Phi'_\beta) \subseteq \mathcal{B}(\Phi'_\beta)$. Therefore, $\mathcal{C}(\Phi'_\beta) \subseteq \mathcal{B}(\Phi'_\beta)$.

A function $\mu : \mathcal{Z}(\Phi') \rightarrow [0, \infty]$ is called a *cylindrical measure* on Φ' , if for each finite subset $M \subseteq \Phi'$ the restriction of μ to $\mathcal{C}(\Phi', M)$ is a measure. A cylindrical measure μ is said to be *finite* if $\mu(\Phi') < \infty$ and a *cylindrical probability measure* if $\mu(\Phi') = 1$. The complex-valued function $\widehat{\mu} : \Phi \rightarrow \mathbb{C}$ defined by

$$\widehat{\mu}(\phi) = \int_{\Phi'} e^{if[\phi]} \mu(df) = \int_{-\infty}^{\infty} e^{iz} \mu_\phi(dz), \quad \forall \phi \in \Phi,$$

where for each $\phi \in \Phi$, $\mu_\phi := \mu \circ \pi_\phi^{-1}$, is called the *characteristic function* of μ . In general, a cylindrical measure on Φ' does not extend to a Borel measure on Φ'_β . However, necessary and sufficient conditions for this can be given in terms of the continuity of its characteristic function by means of the Minlos theorem (see [7], Theorem III.1.3, p.88).

A *cylindrical random variable* in Φ' is a linear map $X : \Phi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$. If $Z = \mathcal{Z}(\phi_1, \dots, \phi_n; A)$ is a cylindrical set, for $\phi_1, \dots, \phi_n \in \Phi$ and $A \in \mathcal{B}(\mathbb{R}^n)$, let

$$\mu_X(Z) := \mathbb{P}((X(\phi_1), \dots, X(\phi_n)) \in A) = \mathbb{P} \circ X^{-1} \circ \pi_{\phi_1, \dots, \phi_n}^{-1}(A).$$

The map μ_X is a cylindrical probability measure on Φ' and it is called the *cylindrical distribution* of X . Conversely, to every cylindrical probability measure μ on Φ' there is a canonical cylindrical random variable for which μ is its cylindrical distribution (see [28], p.256-8).

If X is a cylindrical random variable in Φ' , the *characteristic function* of X is defined to be the characteristic function $\hat{\mu}_X : \Phi \rightarrow \mathbb{C}$ of its cylindrical distribution μ_X . Therefore, $\hat{\mu}_X(\phi) = \mathbb{E}e^{iX(\phi)}$, $\forall \phi \in \Phi$. Also, we say that X is *n-integrable* if $\mathbb{E}(|X(\phi)|^n) < \infty$, $\forall \phi \in \Phi$, and has *zero mean* if $\mathbb{E}(X(\phi)) = 0$, $\forall \phi \in \Phi$.

Let X be a Φ'_β -valued random variable, i.e. $X : \Omega \rightarrow \Phi'_\beta$ is a $\mathcal{F}/\mathcal{B}(\Phi'_\beta)$ -measurable map. We denote by μ_X the distribution of X , i.e. $\mu_X(\Gamma) = \mathbb{P}(X \in \Gamma)$, $\forall \Gamma \in \mathcal{B}(\Phi'_\beta)$, and it is a Borel probability measure on Φ'_β . For each $\phi \in \Phi$ we denote by $X[\phi]$ the real-valued random variable defined by $X[\phi](\omega) := X(\omega)[\phi]$, for all $\omega \in \Omega$. Then, the mapping $\phi \mapsto X[\phi]$ defines a cylindrical random variable. Therefore, the above concepts of characteristic function and integrability can be analogously defined for Φ'_β -valued random variables in terms of the cylindrical random variable they determines.

If X is a cylindrical random variable in Φ' , a Φ'_β -valued random variable Y is called a *version* of X if for every $\phi \in \Phi$, $X(\phi) = Y[\phi]$ \mathbb{P} -a.e. The following results establish alternative characterizations for regular random variables.

A Φ'_β -valued random variable X is called *regular* if there exists a weaker countably Hilbertian topology θ on Φ such that $\mathbb{P}(\omega : X(\omega) \in \Phi'_\theta) = 1$.

Theorem 2.2 ([12], Theorem 2.9). *Let X be a Φ'_β -valued random variable. Consider the statements:*

- (1) *X is regular.*
 - (2) *The map $X : \Phi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$, $\phi \mapsto X[\phi]$ is continuous.*
 - (3) *The distribution μ_X of X is a Radon probability measure.*
- Then, (1) \Leftrightarrow (2) and (2) \Rightarrow (3). Moreover, if Φ is barrelled, we have (3) \Rightarrow (1).*

Let $J = [0, \infty)$ or $J = [0, T]$ for some $T > 0$. We say that $X = \{X_t\}_{t \in J}$ is a *cylindrical process* in Φ' if X_t is a cylindrical random variable, for each $t \in J$. Clearly, any Φ'_β -valued stochastic processes $X = \{X_t\}_{t \in J}$ defines a cylindrical process under the prescription: $X[\phi] = \{X_t[\phi]\}_{t \in J}$, for each $\phi \in \Phi$. We will say that it is the *cylindrical process determined by X* .

A Φ'_β -valued processes $Y = \{Y_t\}_{t \in J}$ is said to be a Φ'_β -valued *version* of the cylindrical process $X = \{X_t\}_{t \in J}$ on Φ' if for each $t \in J$, Y_t is a Φ'_β -valued version of X_t .

Let $X = \{X_t\}_{t \in J}$ be a Φ'_β -valued process. We say that X is *continuous* (respectively *càdlàg*) if for \mathbb{P} -a.e. $\omega \in \Omega$, the *sample paths* $t \mapsto X_t(\omega) \in \Phi'_\beta$ of X are continuous (respectively right-continuous with left limits). On the other hand, we say that X is *regular* if for every $t \in J$, X_t is a regular random variable. The following two results contains some useful properties of Φ'_β -valued regular processes. For proofs see Chapter 1 in [11].

Proposition 2.3. *Let $X = \{X_t\}_{t \in J}$ and $Y = \{Y_t\}_{t \in J}$ be Φ'_β -valued regular stochastic processes such that for each $\phi \in \Phi$, $X[\phi] = \{X_t[\phi]\}_{t \in J}$ is a version of $Y = \{Y_t[\phi]\}_{t \in J}$. Then X is a version of Y . Furthermore, if X and Y are right-continuous then they are indistinguishable processes.*

Proposition 2.4. *Let $X^1 = \{X_t^1\}_{t \in J}$, ..., $X^k = \{X_t^k\}_{t \in J}$ be Φ'_β -valued regular processes. Then, X^1, \dots, X^k are independent if and only if for all $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$, the \mathbb{R}^n -valued processes $\{(X_t^j[\phi_1], \dots, X_t^j[\phi_n]) : t \in J\}$, $j = 1, \dots, k$, are independent.*

The following sequence of results offers an extension of Minlos' theorem to the more general case of cylindrical stochastic processes defined on Φ . Here it is important to remark that equicontinuity of a family of cylindrical random variables is equivalent to equicontinuity at zero of its characteristic functions (see [34], Proposition IV.3.4).

We start with one of the main tools we have at our disposal and that plays a fundamental role throughout this work. It establishes conditions for a cylindrical stochastic process in Φ' to have a regular continuous or càdlàg version.

Theorem 2.5 (Regularization Theorem; [12], Theorem 3.2). *Let $X = \{X_t\}_{t \geq 0}$ be a cylindrical process in Φ' satisfying:*

- (1) For each $\phi \in \Phi$, the real-valued process $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$ has a continuous (respectively càdlàg) version.
- (2) For every $T > 0$, the family $\{X_t : t \in [0, T]\}$ of linear maps from Φ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is equicontinuous.

Then, there exists a countably Hilbertian topology ϑ_X on Φ and a $(\widetilde{\Phi_{\vartheta_X}})'_{\beta}$ -valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t \geq 0}$, such that for every $\phi \in \Phi$, $Y[\phi] = \{Y_t[\phi]\}_{t \geq 0}$ is a version of $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$. Moreover, Y is a Φ'_{β} -valued, regular, continuous (respectively càdlàg) version of X that is unique up to indistinguishable versions.

The following result is a particular case of the regularization theorem that establish conditions for the existence of a regular continuous or càdlàg version with finite moments and taking values in one of the Hilbert spaces Φ'_q .

Theorem 2.6 ([12], Theorem 4.3). *Let $X = \{X_t\}_{t \geq 0}$ be a cylindrical process in Φ' satisfying:*

- (1) *For each $\phi \in \Phi$, the real-valued process $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$ has a continuous (respectively càdlàg) version.*
- (2) *There exists $n \in \mathbb{N}$ and a continuous Hilbertian semi-norm ϱ on Φ such that for all $T > 0$ there exists $C(T) > 0$ such that*

$$\mathbb{E} \left(\sup_{t \in [0, T]} |X_t(\phi)|^n \right) \leq C(T) \varrho(\phi)^n, \quad \forall \phi \in \Phi. \quad (2.3)$$

Then, there exists a continuous Hilbertian semi-norm q on Φ , $\varrho \leq q$, such that $i_{\varrho, q}$ is Hilbert-Schmidt and there exists a Φ'_q -valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t \geq 0}$, satisfying:

- (a) For every $\phi \in \Phi$, $Y[\phi] = \{Y_t[\phi]\}_{t \geq 0}$ is a version of $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$,
- (b) For every $T > 0$, $\mathbb{E} \left(\sup_{t \in [0, T]} q'(Y_t)^n \right) < \infty$.

Furthermore, Y is a Φ'_{β} -valued continuous (respectively càdlàg) version of X that is unique up to indistinguishable versions.

The following is a converse of the regularization theorem when Φ is a barrelled nuclear space.

Theorem 2.7. *Let Φ be a barrelled nuclear space and $L = \{L_t\}_{t \geq 0}$, be a cylindrical process in Φ' . Suppose that for every $t \geq 0$ the cylindrical probability distribution of L_t can be extended to a Radon probability measure μ_{L_t} on Φ'_{β} such that for every $T > 0$ the family $\{\mu_{L_t} : t \in [0, T]\}$ is uniformly tight. Then, for every $T > 0$ the family of linear maps $\{L_t : t \in [0, T]\}$ is equicontinuous.*

Proof. Let $T > 0$ and $\epsilon > 0$. First, because the family $\{\mu_{L_t} : t \in [0, T]\}$ is uniformly tight, there exists a compact $K \subseteq \Phi'_{\beta}$ such that $\mu_{L_t}(K^c) < \epsilon$ for all $t \in [0, T]$.

Now, as K is compact and hence bounded in Φ'_{β} (recall Φ'_{β} is Hausdorff), and because Φ is barrelled, then K is a equicontinuous subset of Φ' (see [27], Theorem IV.5.2, p.141) and consequently the polar K^0 of K is a neighborhood of zero of Φ (see [20], Theorem 8.6.4(b), p.246). But as Φ is nuclear, there exists a continuous Hilbertian semi-norm p on Φ such that $B_p(1/\epsilon) \subseteq K^0$. Therefore, from the properties of polar sets (see [20], Chap.8) we have that $K \subseteq (K^0)^0 \subseteq B_{p'}(\epsilon) := \{f \in \Phi' : p'(f) = \sup_{\phi \in B_p(1)} |f[\phi]| \leq \epsilon\} = B_p(1/\epsilon)^0$. Thus, $B_{p'}(\epsilon)^c \subseteq K^c$.

On the other hand, note that for every $\phi \in B_p(1)$ we have $\pi_{\phi}^{-1}([-\epsilon, \epsilon]^c) = \{f \in \Phi' : |f[\phi]| > \epsilon\} \subseteq B_{p'}(\epsilon)^c = \{f \in \Phi' : p'(f) = \sup_{\phi \in B_p(1)} |f[\phi]| > \epsilon\}$.

Hence, for every $\phi \in B_p(1)$ it follows from the arguments on the above paragraphs and from the fact that μ_{L_t} is an extension of the cylindrical distribution of L_t that

$$\mathbb{P}(|L_t(\phi)| > \epsilon) = \mathbb{P}(L_t(\phi) \in [-\epsilon, \epsilon]^c) = \mu_{L_t} \circ \pi_{\phi}^{-1}([-\epsilon, \epsilon]^c) \leq \mu_{L_t}(B_{p'}(\epsilon)^c) \leq \mu_{L_t}(K^c) < \epsilon,$$

for all $t \in [0, T]$. But because $B_p(1)$ is a neighborhood of zero of Φ , the above shows that the family of linear maps $\{L_t : t \in [0, T]\}$ is equicontinuous at zero, and hence equicontinuous. \square

3 Lévy Processes and Infinitely Divisible Measures.

In this section we study the relationship between Lévy processes and infinitely divisible measures. The link between these two concepts are the cylindrical Lévy processes and the semigroups of probability measures.

3.1 Infinitely Divisible Measures and Convolution Semigroups in the Strong Dual.

Let Ψ be a locally convex space. A measure $\mu \in \mathfrak{M}_R^1(\Psi'_\beta)$ is called *infinitely divisible* if for every $n \in \mathbb{N}$ there exist a n -th root of μ , i.e. a measure $\mu_n \in \mathfrak{M}_R^1(\Psi'_\beta)$ such that $\mu = \mu_n^{*n}$. We denote by $\mathfrak{I}(\Psi'_\beta)$ the set of all infinitely divisible measures on Ψ'_β .

A family $\{\mu_t\}_{t \geq 0} \subseteq \mathfrak{M}_R^1(\Psi'_\beta)$ is said to be a *convolution semigroup* if $\mu_s * \mu_t = \mu_{s+t}$ for any $s, t \geq 0$ and $\mu_0 = \delta_0$. Moreover, we say that the convolution semigroup is *continuous* if the mapping $t \mapsto \mu_t$ from $[0, \infty)$ into $\mathfrak{M}_R^1(\Psi'_\beta)$ is continuous in the weak topology.

The following result follows easily from the definition of continuous convolution semigroup.

Proposition 3.1. *If $\{\mu_t\}_{t \geq 0}$ is a convolution semigroup in $\mathfrak{M}_R^1(\Psi'_\beta)$, then $\forall t \geq 0$ $\mu_t \in \mathfrak{I}(\Psi'_\beta)$.*

Now, to prove the converse of Proposition 3.1 we will need the following definitions.

Let μ be an infinitely divisible measure on Ψ'_β . We define the *root set* of μ by

$$R(\mu) := \bigcup_{n \geq 1} \{ \nu^n : \nu \in \mathfrak{M}_R^1(\Psi'_\beta) \text{ with } \nu^n = \mu, 1 \leq n \leq \infty \}.$$

We say that μ is *root compact* if its root set $R(\mu)$ is uniformly tight.

We are ready for the main result of this section. As stated on its proof, the main arguments are based on several results due to E. Siebert (see [29, 30]).

Theorem 3.2. *Assume that Ψ is a locally convex space for which Ψ'_β is quasi-complete. If $\mu \in \mathfrak{I}(\Psi'_\beta)$, then there exists a unique continuous convolution semigroup $\{\mu_t\}_{t \geq 0}$ in $\mathfrak{M}_R^1(\Psi'_\beta)$ such that $\mu_1 = \mu$.*

Proof. First, as Ψ'_β is locally convex and $\mu \in \mathfrak{I}(\Psi'_\beta)$, there exists a rational continuous convolution semigroup $\{\nu_t\}_{t \in \mathbb{Q} \cap [0, \infty)}$ in $\mathfrak{M}_R^1(\Psi'_\beta)$ such that $\nu_1 = \mu$ (see [29], Korollar 5.4).

Now, as $\mu = \nu_1 = \nu_{1/q}^{*q}$, then $\nu_{1/q}$ is a root of μ for each $q \in \mathbb{N} \setminus \{0\}$. But as for $p, q \in \mathbb{N} \setminus \{0\}$ we have $\nu^{*p/q} = \nu_{1/q}^{*p}$, then we have that $\nu_t \in R(\mu)$ for each $t \in \mathbb{Q} \cap [0, 1]$.

On the other hand, as μ is tight (is Radon) and Ψ'_β is a quasi-complete locally convex space, the root set $R(\mu)$ of μ is uniformly tight (see [29], Satz 6.2 and 6.4). Hence, the set $\{\nu_t\}_{t \in \mathbb{Q} \cap [0, 1]}$ is uniformly tight and by Prokhorov's theorem it is relatively compact. This last property guarantees the existence of a (unique) continuous convolution semigroup $\{\mu_t\}_{t \geq 0}$ in $\mathfrak{M}_R^1(\Psi'_\beta)$ such that $\nu_t = \mu_t$ for each $t \in \mathbb{Q} \cap [0, \infty)$ (see [30], Proposition 5.3). Therefore, $\mu = \nu_1 = \mu_1$. \square

The following result will be of great importance in further developments.

Lemma 3.3. *Assume that Ψ'_β is quasi-complete and let $\{\mu_t\}_{t \geq 0}$ be a continuous convolution semigroup in $\mathfrak{M}_R^1(\Psi'_\beta)$. Then, $\forall T > 0$ $\{\mu_t : t \in [0, T]\}$ is uniformly tight.*

Proof. Let $T > 0$. Similar arguments to those used in the proof of Theorem 3.2 shows that $\{\mu_t\}_{t \in \mathbb{Q} \cap [0, T]} \subseteq R(\mu_T)$, and because μ_T is tight, the root set $R(\mu)$ of μ is uniformly tight (see [29], Satz 6.2 and 6.4), and hence $\{\mu_t\}_{t \in \mathbb{Q} \cap [0, T]}$ is also uniformly tight.

Now, note that for each $r \in \mathbb{I} \cup [0, T]$ the continuity of the semigroup $\{\mu_t\}_{t \geq 0}$ shows that $\mu_r = \lim_{q \searrow r, q \in \mathbb{Q} \cap [0, T]} \mu_q$ in the weak topology. Therefore, $\{\mu_t\}_{t \in [0, T]}$ is in the weak closure $\overline{\{\mu_t\}_{t \in \mathbb{Q} \cap [0, T]}}$ of $\{\mu_t\}_{t \in \mathbb{Q} \cap [0, T]}$. But because the weak closure of an uniformly tight family in $\mathfrak{M}_R^1(\Psi'_\beta)$ is also uniformly tight (see [34], Theorem I.3.5), then it follows that $\overline{\{\mu_t\}_{t \in \mathbb{Q} \cap [0, T]}}$ is uniformly tight and hence $\{\mu_t\}_{t \in [0, T]}$ is uniformly tight too. \square

3.2 Lévy Processes and Cylindrical Lévy Processes

From now on and unless otherwise specified, Φ will always be a nuclear space over \mathbb{R} .

We start with our definition of Lévy processes on the dual of a nuclear space.

Definition 3.4. A Φ'_β -valued process $L = \{L_t\}_{t \geq 0}$ is called a *Lévy process* if it satisfies:

- (1) $L_0 = 0$ a.s.
- (2) L has *independent increments*, i.e. for any $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \dots < t_n < \infty$ the Φ'_β -valued random variables $L_{t_1}, L_{t_2} - L_{t_1}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent.
- (3) L has *stationary increments*, i.e. for any $0 \leq s \leq t$, $L_t - L_s$ and L_{t-s} are identically distributed.
- (4) For every $t \geq 0$ the distribution μ_t of L_t is a Radon measure and the mapping $t \mapsto \mu_t$ from $[0, \infty)$ into $\mathfrak{M}_R^1(\Phi'_\beta)$ is continuous at 0 in the weak topology.

The probability distributions of a Φ'_β -valued Lévy process satisfy the following properties:

Theorem 3.5. If $L = \{L_t\}_{t \geq 0}$ is a Lévy process in Φ'_β , the family of probability distributions $\{\mu_{L_t}\}_{t \geq 0}$ of L is a continuous convolution semigroup in $\mathfrak{M}_R^1(\Phi'_\beta)$. Moreover, each μ_{L_t} is infinitely divisible for every $t \geq 0$. Furthermore, if Φ is also a barrelled space, then for each $T > 0$ the family $\{\mu_{L_t} : t \in [0, T]\}$ is uniformly tight.

Proof. The semigroup property of $\{\mu_{L_t}\}_{t \geq 0}$ is an easy consequence of the stationary and independent increments properties of L . The weak continuity is part of our definition of Lévy process. The fact that each μ_{L_t} is infinitely divisible follows from Proposition 3.1. Finally, if Φ is also a barrelled space, then Φ'_β is quasi-complete (see [27], Theorem IV.6.1, p.148). Hence, the uniform tightness of $\{\mu_{L_t} : t \in [0, T]\}$ for each $T > 0$ follows from Lemma 3.3. \square

Following the definition given in Applebaum and Riedle [2] for cylindrical Lévy processes in Banach spaces, we introduce the following definition.

Definition 3.6. A cylindrical process $L = \{L_t\}_{t \geq 0}$ in Φ' is said to be a *cylindrical Lévy process* if $\forall n \in \mathbb{N}$, $\phi_1, \dots, \phi_n \in \Phi$, the \mathbb{R}^n -valued process $\{(L_t(\phi_1), \dots, L_t(\phi_n))\}_{t \geq 0}$ is a Lévy process.

Lemma 3.7. Every Φ'_β -valued Lévy process $L = \{L_t\}_{t \geq 0}$ determines a cylindrical Lévy process in Φ' .

Proof. Let $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$. It is clear that $(L_0[\phi_1], \dots, L_0[\phi_n]) = 0$ \mathbb{P} -a.e. The fact that $\{(L_t[\phi_1], \dots, L_t[\phi_n])\}_{t \geq 0}$ has stationary and independent increments follows from the corresponding properties of L as a Φ'_β -valued process (see Proposition 2.4). Finally, the stochastic continuity of $\{(L_t[\phi_1], \dots, L_t[\phi_n])\}_{t \geq 0}$ is a consequence of the weak continuity of the map $t \mapsto \mu_t$ (see [1], Proposition 1.4.1). \square

The following result is a converse of Lemma 3.7.

Theorem 3.8. Let $L = \{L_t\}_{t \geq 0}$ be a cylindrical Lévy process in Φ' such that for every $T > 0$, the family $\{L_t : t \in [0, T]\}$ of linear maps from Φ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is equicontinuous. Then, there exists a countably Hilbertian topology ϑ_L on Φ and a $(\widetilde{\Phi_{\vartheta_L}})'_\beta$ -valued càdlàg process $Y = \{Y_t\}_{t \geq 0}$, such that for every $\phi \in \Phi$, $Y[\phi] = \{Y_t[\phi]\}_{t \geq 0}$ is a version of $L(\phi) = \{L_t(\phi)\}_{t \geq 0}$. Moreover, Y is a Φ'_β -valued, regular, càdlàg Lévy process that is a version of L and that is unique up to indistinguishable versions.

Proof. First, as for each $\phi \in \Phi$ the real-valued process $L(\phi) = \{L_t(\phi)\}_{t \geq 0}$ is a Lévy process, then it has a càdlàg version (see Theorem 2.1.8 of Applebaum [1], p.87). Hence, L satisfies all the conditions of the regularization theorem (Theorem 2.5) and this theorem shows the existence of a countably Hilbertian topology ϑ_L on Φ and a $(\widetilde{\Phi_{\vartheta_L}})'_\beta$ -valued càdlàg process $Y = \{Y_t\}_{t \geq 0}$, such that for every $\phi \in \Phi$, $Y[\phi] = \{Y_t[\phi]\}_{t \geq 0}$ is a version of $L(\phi) = \{L_t(\phi)\}_{t \geq 0}$. Moreover, it is a consequence of the regularization theorem that Y is a Φ'_β -valued, regular, càdlàg version of L that is unique up to indistinguishable versions.

Our next step is to show that Y is a Φ'_β -valued Lévy process. First, as $Y_0[\phi] = L_0(\phi) = 0$ \mathbb{P} -a.e. for every $\phi \in \Phi$, it follows that $Y_0 = 0$ \mathbb{P} -a.e. (Proposition 2.3). Second, as for each

$\phi_1, \dots, \phi_n \in \Phi$, the \mathbb{R}^n -valued process $\{(L_t(\phi_1), \dots, L_t(\phi_n))\}_{t \geq 0}$ has independent and stationary increments, and because for each $t \geq 0$ we have that

$$(L_t(\phi_1), \dots, L_t(\phi_n)) = (Y_t[\phi_1], \dots, Y_t[\phi_n]), \quad \mathbb{P} - \text{a.e.},$$

then the \mathbb{R}^n -valued process $\{(Y_t[\phi_1], \dots, Y_t[\phi_n])\}_{t \geq 0}$ also has independent and stationary increments for every $\phi_1, \dots, \phi_n \in \Phi$. Hence, because Y is a Φ'_β -valued regular process, it then follows from Propositions 2.3 and 2.4 that Y has independent and stationary increments.

Now, the fact that Y is a Φ'_β -valued regular process and Theorem 2.2 shows that for each $t \geq 0$ the probability distribution μ_t of Y_t is a Radon measure.

Our final step to show that Y is a Φ'_β -valued Lévy process is to prove that the mapping $t \mapsto \mu_t$ from $[0, \infty)$ into $\mathfrak{M}_R^1(\Phi'_\beta)$ is continuous in the weak topology.

Let $t \geq 0$. Our objective is to show that for any net $\{s_\alpha\}$ in $[0, \infty)$ such that $\lim_\alpha s_\alpha = t$ we have $\lim_\alpha \mu_{s_\alpha} = \mu_t$ in the weak topology on $\mathfrak{M}_R^1(\Phi'_\beta)$. As convergence of filterbases is only determined by terminal sets, we can choose without loss of generality some sufficiently large $T > 0$ and consider only nets in $[0, T]$ satisfying $\{s_\alpha\}$ such that $\lim_\alpha s_\alpha = t$. Let $\{s_\alpha\}$ be such a net.

First, as for each $\phi \in \Phi$, $Y[\phi] = \{Y_t[\phi]\}_{t \geq 0}$ is stochastically continuous, it follows that the family $\{Y_{s_\alpha}[\phi]\}$ converges in probability to $Y_t[\phi]$. Now, this last property in turns shows that $\lim_\alpha \hat{\mu}_{s_\alpha}(\phi) = \hat{\mu}_t(\phi)$ for every $\phi \in \Phi$.

Now, for each $r \geq 0$ denote by ν_r the cylindrical distribution of the cylindrical random variable L_r . Then, the equicontinuity of the family $\{L_r : r \in [0, T]\}$ of linear maps from Φ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ implies that the family of characteristic functions $\{\hat{\nu}_r\}_{r \in [0, T]}$ is equicontinuous at zero. But as for each $r \geq 0$, $\hat{\nu}_r(\phi) = \hat{\mu}_r(\phi)$ for all $\phi \in \Phi$, we then have that the family of characteristic functions $\{\hat{\mu}_r\}_{r \in [0, T]}$ of $\{Y_r\}_{r \in [0, T]}$ is equicontinuous at zero. However, as Φ is a nuclear space the equicontinuity of $\{\hat{\mu}_r\}_{r \in [0, T]}$ at zero implies that $\{\mu_r\}_{r \in [0, T]}$ is uniformly tight (see [7], Lemma III.2.3, p.103-4). This last in turn shows that $\{\mu_{s_\alpha}\}$ is uniformly tight, and by the Prokhorov's theorem (see [7], Theorem III.2.1, p.98) the family $\{\mu_{s_\alpha}\}$ is relatively compact in the weak topology. Because we also have that $\lim_\alpha \hat{\mu}_{s_\alpha}(\phi) = \hat{\mu}_t(\phi)$ for every $\phi \in \Phi$, we then conclude that $\lim_\alpha \mu_{s_\alpha} = \mu_t$ in the weak topology (see [34], Theorem IV.3.1, p.224-5). Consequently, the map $t \mapsto \mu_t$ is continuous in the weak topology and Y is a Φ'_β -valued Lévy process. \square

An important variation of the above theorem is the following:

Theorem 3.9. *Let $L = \{L_t\}_{t \geq 0}$ be a cylindrical Lévy process in Φ' . Assume that there exist $n \in \mathbb{N}$ and a continuous Hilbertian semi-norm ϱ on Φ such that for all $T > 0$ there is a $C(T) > 0$ such that*

$$\mathbb{E} \left(\sup_{t \in [0, T]} |L_t(\phi)|^n \right) \leq C(T) \varrho(\phi)^n, \quad \forall \phi \in \Phi.$$

Then, there exists a continuous Hilbertian semi-norm q on Φ , $\varrho \leq q$, such that $i_{\varrho, q}$ is Hilbert-Schmidt and there exists a Φ'_q -valued càdlàg Lévy process $Y = \{Y_t\}_{t \geq 0}$, satisfying:

- (a) *For every $\phi \in \Phi$, $Y[\phi] = \{Y_t[\phi]\}_{t \geq 0}$ is a version of $L(\phi) = \{L_t(\phi)\}_{t \geq 0}$,*
- (b) *For every $T > 0$, $\mathbb{E} \left(\sup_{t \in [0, T]} q'(Y_t)^n \right) < \infty$.*

Moreover, Y is a Φ'_β -valued, regular, càdlàg version of L that is unique up to indistinguishable versions. Furthermore, if the real-valued process $L(\phi)$ is continuous for each $\phi \in \Phi$, then Y can be choose to be continuous in Φ'_q and hence in Φ'_β .

Proof. The existence of the Φ'_q -valued càdlàg process $Y = \{Y_t\}_{t \geq 0}$ satisfying the conditions in the statement of the theorem follows from Theorem 2.6. Finally, similar arguments to those used in the proof of Theorem 3.8 show that Y is a Φ'_q -valued Lévy process. \square

We now provide a sufficient condition for the existence of a càdlàg version for a Φ'_β -valued Lévy process.

Theorem 3.10. *Let $L = \{L_t\}_{t \geq 0}$ be a Φ'_β -valued Lévy process. Suppose that for every $T > 0$, the family $\{L_t : t \in [0, T]\}$ of linear maps from Φ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ given by $\phi \mapsto L_t[\phi]$ is*

equicontinuous. Then, L has a Φ'_β -valued, regular, càdlàg version $\tilde{L} = \{\tilde{L}_t\}_{t \geq 0}$ that is also a Lévy process. Moreover, there exists a countably Hilbertian topology ϑ_L on Φ such that \tilde{L} is a $(\Phi_{\vartheta_L})'_\beta$ -valued càdlàg process.

Proof. First, note that our assumption on L implies that L is regular. This is because for each $t \geq 0$ the fact that $L_t : \Phi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous shows that L_t is a regular random variable in Φ'_β (Theorem 2.2).

Now, as L is a Φ'_β -valued Lévy process the cylindrical process determined by L is a cylindrical Lévy process (Lemma 3.7). But from our assumptions on L , this cylindrical Lévy process satisfies the assumptions in Theorem 3.8. Therefore, there exists a Φ'_β -valued, regular, càdlàg Lévy process $\tilde{L} = \{\tilde{L}_t\}_{t \geq 0}$, such that for every $\phi \in \Phi$, $\tilde{L}_t[\phi] = L_t[\phi]$ \mathbb{P} -a.e. for each $t \geq 0$. This last property together with the fact that both \tilde{L} and L are regular process shows that \tilde{L} is a version of L (Proposition 2.3). Finally, from Theorem 3.8 there exists a countably Hilbertian topology ϑ_L on Φ such that \tilde{L} is a $(\Phi_{\vartheta_L})'_\beta$ -valued càdlàg process. \square

Corollary 3.11. *If Φ is a barrelled nuclear space and $L = \{L_t\}_{t \geq 0}$ is a Φ'_β -valued Lévy process, then L has a Φ'_β -valued càdlàg version satisfying the properties given in Theorem 3.10.*

Proof. It follows from Theorem 3.5 that for every $T > 0$ the family $\{\mu_{L_t} : t \in [0, T]\}$ is uniformly tight. Then, it follows from Theorem 2.7 that L satisfies the assumptions on Theorem 3.10. Hence, the result follows. \square

Finally, the next result provides sufficient conditions for the existence of a càdlàg version that is a Lévy process with finite n -th moment in some of the Hilbert spaces Φ'_q .

Theorem 3.12. *Let $L = \{L_t\}_{t \geq 0}$ be a Φ'_β -valued Lévy process. Assume that there exist $n \in \mathbb{N}$ and a continuous Hilbertian semi-norm ϱ on Φ such that for all $T > 0$ there is a $C(T) > 0$ such that*

$$\mathbb{E} \left(\sup_{t \in [0, T]} |L_t[\phi]|^n \right) \leq C(T) \varrho(\phi)^n, \quad \forall \phi \in \Phi.$$

Then, there exists a continuous Hilbertian semi-norm q on Φ , $\varrho \leq q$, such that $i_{\varrho, q}$ is Hilbert-Schmidt and a Φ'_q -valued, càdlàg (continuous if L is continuous), Lévy process $\tilde{L} = \{\tilde{L}_t\}_{t \geq 0}$ that is a version of L . Moreover, $\mathbb{E} \left(\sup_{t \in [0, T]} q'(Y_t)^n \right) < \infty \forall T > 0$.

Proof. The proof follows from Theorem 3.9 and similar arguments to those used in the proof of Theorem 3.10. \square

3.3 Correspondence of Lévy Processes and Infinitely Divisible Measures

We have already show in Theorem 3.5 that for every Φ'_β -valued Lévy process $L = \{L_t\}_{t \geq 0}$ the probability distribution μ_{L_t} of L_t is infinitely divisible for each $t \geq 0$. In this section we will show that if the space Φ is barrelled and nuclear, to every infinitely divisible measure μ on Φ'_β there corresponds a Φ'_β -valued Lévy process L such that $\mu_{L_1} = \mu$.

In order to prove our main result (Theorem 3.14), we will need the following theorem that establishes the existence of a cylindrical Lévy process from a given family of cylindrical probability measures with some semigroup properties. We formulate our result in the more general context of Hausdorff locally convex spaces. The definitions of cylindrical probability measure and cylindrical Lévy process are exactly the same to those given in Sections 2.2 and 3.2.

Theorem 3.13. *Let Ψ be a Hausdorff locally convex space. Let $\{\mu_t\}_{t \geq 0}$ be a family of cylindrical measures on Ψ' such that for every finite collection $\psi_1, \psi_2, \dots, \psi_n \in \Psi$, the family $\{\mu_t \circ \pi_{\psi_1, \psi_2, \dots, \psi_n}^{-1}\}_{t \geq 0}$ is a continuous convolution semigroup of probability measures on \mathbb{R}^n . Then, there exists a cylindrical process $L = \{L_t\}_{t \geq 0}$ in Ψ' defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that:*

(1) *For every $t \geq 0$, $\psi_1, \psi_2, \dots, \psi_n \in \Psi$ and $\Gamma \in \mathcal{B}(\mathbb{R}^n)$,*

$$\mathbb{P}((L_t(\psi_1), L_t(\psi_2), \dots, L_t(\psi_n)) \in \Gamma) = \mu_t \circ \pi_{\psi_1, \psi_2, \dots, \psi_n}^{-1}(\Gamma).$$

(2) L is a cylindrical Lévy process in Ψ' .

For the proof of Theorem 3.13 we will need to deal with projective systems of measure spaces. For the convenience of the reader we recall their definition (see [24] p.17-19 for more details).

Let $\{(\Omega_\alpha, \Sigma_\alpha, P_\alpha) : \alpha \in D\}$ be a family of measure spaces, where D is a directed set, and let $\{g_{\alpha\beta}, \alpha < \beta, \alpha, \beta \in D\}$ be a family of mappings such that: (i) $g_{\alpha\beta} : \Omega_\beta \rightarrow \Omega_\alpha$, and $g_{\alpha\beta}^{-1}(\Sigma_\alpha) \subseteq \Sigma_\beta$, (ii) for any $\alpha < \beta < \gamma$, $g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma}$ and $g_{\alpha\alpha} = \text{identity}$, and (iii) for every $\alpha < \beta$, $P_\alpha = P_\beta \circ g_{\alpha\beta}^{-1}$. Then, the abstract collection $\{(\Omega_\alpha, \Sigma_\alpha, P_\alpha, g_{\alpha\beta})_{\alpha < \beta} : \alpha, \beta \in D\}$ is called a projective systems of measure spaces (of Hausdorff topological spaces if each $(\Omega_\alpha, \Sigma_\alpha)$ is a Hausdorff topological space, the measure P_α is regular in the measure theory sense and each $g_{\alpha\beta}$ is continuous).

Proof of Theorem 3.13. Our first objective is to define a projective system of Hausdorff topological spaces for which the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be its projective limit (see [24]).

Let \mathbb{F} be the set of all finite collection of elements of Ψ . For any $F = (\psi_1, \psi_2, \dots, \psi_n) \in \mathbb{F}$, define $\pi_F := \pi_{\psi_1, \psi_2, \dots, \psi_n}$, where recall that $\pi_{\psi_1, \psi_2, \dots, \psi_n}(f) = (f[\psi_1], f[\psi_2], \dots, f[\psi_n])$, for all $f \in \Psi'$. Then, it is clear that the map $\pi_F : \Psi' \mapsto \Omega^F$ is continuous, where $\Omega^F := \mathbb{R}^n$.

Now, for $F \in \mathbb{F}$, define $\mu_t^F := \mu_t \circ \pi_F^{-1}$, for all $t \geq 0$. Then, from our assumptions on $\{\mu_t\}_{t \geq 0}$ we have that $\{\mu_t^F\}_{t \geq 0}$ is a continuous convolution semigroup of probability measures on Ω^F .

Consider on \mathbb{F} the partial order $\leq_{\mathbb{F}}$ determined by the set inclusion. For any $F, G \in \mathbb{F}$ satisfying $F \leq_{\mathbb{F}} G$, denote by $g_{F,G} : \Omega^G \rightarrow \Omega^F$ the canonical projection from Ω^G into Ω^F . For any $F, G, H \in \mathbb{F}$ satisfying $F \leq_{\mathbb{F}} G \leq_{\mathbb{F}} H$, it follows from the definitions above that we have:

$$g_{F,H} = g_{F,G} \circ g_{G,H}, \quad g_{F,F} = \text{identity on } \Omega^F, \quad (3.1)$$

$$\mu_t^F = \mu_t^G \circ g_{F,G}^{-1}. \quad (3.2)$$

Now, let $\mathbb{A} = \{\{(t_i, \psi_i)\}_{i=1}^n : n \in \mathbb{N}, 0 \leq t_1 \leq t_2 \leq \dots \leq t_n, \psi_1, \psi_2, \dots, \psi_n \in \Psi\}$. Then, $(\mathbb{A}, \leq_{\mathbb{A}})$ is a directed set when $\leq_{\mathbb{A}}$ is the partial order on \mathbb{A} defined as follows:

for $A = \{(t_i, \psi_i)\}_{i=1}^n, B = \{(s_j, \phi_j)\}_{j=1}^m \in \mathbb{A}$, we say $A \leq_{\mathbb{A}} B$ if

$$\{t_1, t_2, \dots, t_n\} \subseteq \{s_1, s_2, \dots, s_m\} \quad \text{and} \quad F = (\psi_1, \psi_2, \dots, \psi_n) \leq_{\mathbb{F}} G = (\phi_1, \phi_2, \dots, \phi_m).$$

For $A \in \mathbb{A}$ as above, define $\Omega^A = \Omega_{t_1}^F \times \Omega_{t_2}^F \times \dots \times \Omega_{t_n}^F$, where $\Omega_{t_i}^F := \Omega^F$ for $i = 1, \dots, n$. Similarly, for $B \in \mathbb{A}$ as above define $\Omega^B = \Omega_{s_1}^G \times \Omega_{s_2}^G \times \dots \times \Omega_{s_m}^G$, with $\Omega_{s_j}^G := \Omega^G$ for $j = 1, \dots, m$. Clearly, Ω^A and Ω^B are Hausdorff topological vector spaces.

Now, note that if $A \leq_{\mathbb{A}} B$, then from the definition of $\leq_{\mathbb{A}}$ we have $\{t_i\}_{i=1}^n \subseteq \{s_j\}_{j=1}^m$. Let s_{j_1}, \dots, s_{j_n} given by $s_{j_i} = t_i$, for $i = 1, \dots, n$. Define the projection $g_{A,B} : \Omega^B \rightarrow \Omega^A$ by the prescription:

$$\begin{aligned} (w_{s_1}, w_{s_2}, \dots, w_{s_m}) \in \Omega^B &\mapsto (g_{F,G}(w_{s_{j_1}}), g_{F,G}(w_{s_{j_2}}), \dots, g_{F,G}(w_{s_{j_n}})) \\ &= (g_{F,G}(w_{t_1}), g_{F,G}(w_{t_2}), \dots, g_{F,G}(w_{t_n})) \in \Omega^A. \end{aligned} \quad (3.3)$$

If $C = \{(r_k, \varphi_k)\}_{k=1}^p \in \mathbb{A}$ is such that $A \leq_{\mathbb{A}} B \leq_{\mathbb{A}} C$, and if we take $H = (\varphi_1, \varphi_2, \dots, \varphi_p) \in \mathbb{F}$, then it is clear from (3.1) that:

$$g_{A,C} = g_{A,B} \circ g_{B,C}, \quad g_{A,A} = \text{identity on } \Omega^A, \quad (3.4)$$

Now, for $A \in \mathbb{A}$ as above, define μ_A by

$$\mu_A(\Gamma_1 \times \dots \times \Gamma_n) = \int_{\Gamma_1} \mu_{t_1}^F(dw_1) \int_{\Gamma_2} \mu_{t_2-t_1}^F(dw_2 - w_1) \dots \int_{\Gamma_n} \mu_{t_n-t_{n-1}}^F(dw_n - w_{n-1}), \quad (3.5)$$

for $\Gamma_i \in \mathcal{B}(\Omega_{t_i}^F)$, $\forall i = 1, \dots, n$. Then μ_A can be extended to a unique measure on Ω^A .

Now, let $\tilde{\Gamma}_i \in \mathcal{B}(\Omega_{t_i}^F)$, $\forall i = 1, \dots, n$. From (3.3) it follows that for $A \leq_{\mathbb{A}} B$ we have:

$$g_{A,B}^{-1}(\Gamma_1 \times \dots \times \Gamma_n) = \Sigma_1 \times \dots \times \Sigma_m, \quad \text{where} \quad \Sigma_j = \begin{cases} \Omega_{s_j}^G, & \text{if } s_j \notin \{s_{j_1}, \dots, s_{j_n}\}, \\ g_{F,G}^{-1}(\tilde{\Gamma}_j), & \text{if } s_j \in \{s_{j_1}, \dots, s_{j_n}\}. \end{cases} \quad (3.6)$$

Hence, from (3.2), (3.5) and (3.6), it follows that:

$$\begin{aligned}
& \mu_B(g_{A,B}^{-1}(\Gamma_1 \times \cdots \times \Gamma_n)) \\
&= \mu_B(\Sigma_1 \times \cdots \times \Sigma_m) \\
&= \int_{g_{F,G}^{-1}(\Gamma_1)} \mu_{s_{j1}}^G(dw_1) \int_{g_{F,G}^{-1}(\Gamma_2)} \mu_{s_{j2}-s_{j1}}^G(dw_2 - w_1) \cdots \int_{g_{F,G}^{-1}(\Gamma_n)} \mu_{s_{jn}-s_{j_{n-1}}}^G(dw_n - w_{n-1}) \\
&= \int_{\Gamma_1} \mu_{t_1}^G \circ g_{F,G}^{-1}(dw_1) \int_{\Gamma_2} \mu_{t_2-t_1}^G \circ g_{F,G}^{-1}(dw_2 - w_1) \cdots \int_{\Gamma_n} \mu_{t_n-t_{n-1}}^G \circ g_{F,G}^{-1}(dw_n - w_{n-1}) \\
&= \int_{\Gamma_1} \mu_{t_1}^F(dw_1) \int_{\Gamma_2} \mu_{t_2-t_1}^F(dw_2 - w_1) \cdots \int_{\Gamma_n} \mu_{t_n-t_{n-1}}^F(dw_n - w_{n-1}) \\
&= \mu_A(\Gamma_1 \times \cdots \times \Gamma_n).
\end{aligned} \tag{3.7}$$

where on the passage from the first to the second line we used that $\{\mu_t^G\}_{t \geq 0}$ is a convolution semigroup of probability measures on Ω^G . Then, from a standard argument it follows that (3.7) extends to

$$\mu_B \circ g_{A,B}^{-1} = \mu_A, \quad \forall A, B \in \mathbb{A}, A \leq_{\mathbb{A}} B. \tag{3.8}$$

We then conclude that $\{(\Omega^A, \mathcal{B}(\Omega^A), \mu_A, g_{A,B})_{A \leq_{\mathbb{A}} B} : A, B \in \mathbb{A}\}$ is a projective system of Hausdorff topological vector spaces. Hence, from a generalization of the Kolmogorov's Extension Theorem (see [24], Theorem 1.3.4, p.20-1), the latter system admits a unique limit $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \mathbb{R}^{\mathbb{A}} \cong \varprojlim (\Omega^A, g_{A,B})$, $\mathcal{F} = \sigma(\bigcup_{A \in \mathbb{A}} g_A^{-1}(\mathcal{B}(\Omega^A)))$ and $\mathbb{P} = \varprojlim \mu_A$, where $g_A : \Omega \rightarrow \Omega^A$ is the canonical projection determined by the projections $g_{A,B}$.

On the above, $\varprojlim (\Omega^A, g_{A,B})$ is the subset of $\times_{A \in \mathbb{A}} \Omega^A$ of all the elements $(\omega_A)_{A \in \mathbb{A}}$ such that for $A \leq_{\mathbb{A}} B$ we have $g_{A,B}(\omega_B) = \omega_A$. On the other hand, g_A is the projection $(\omega_A)_{A \in \mathbb{A}} \mapsto \omega_A \in \Omega^A$. Also, $\mathbb{P} = \varprojlim \mu_A$ means that \mathbb{P} is a (probability) measure on Ω that satisfies

$$\mu_A = \mathbb{P} \circ g_A^{-1}, \quad \forall A \in \mathbb{A}. \tag{3.9}$$

Our next step is to define a cylindrical process $L = \{L_t\}_{t \geq 0}$ in Ψ' defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the conditions (1) and (2) on the statement of the theorem.

First, it is clear that Ω can be embedded in $\mathbb{R}^{\mathbb{R}_+ \times \Psi} = \times_{(t,\psi)} \mathbb{R}^{(t,\psi)}$, where $\mathbb{R}^{(t,\psi)} = \mathbb{R}$ for each $(t, \psi) \in \mathbb{R}_+ \times \Psi$. This is an easy consequence of the fact that \mathbb{A} consist of finite collections of elements of $\mathbb{R}_+ \times \Psi$.

Now, let $\tilde{I} : \mathbb{R}^{\mathbb{R}_+ \times \Psi} \rightarrow \mathbb{R}^{\mathbb{R}_+ \times \Psi}$ be the identity mapping. Define $\tilde{L} : \mathbb{R}_+ \times \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ by $\tilde{L}(t, \psi) = g_{(t,\psi)} \circ \tilde{I}$, where $g_{(t,\psi)} : \Omega \rightarrow \mathbb{R}$ is the coordinate projection. Then, it follows from the definition of \tilde{L} that:

$$\tilde{L}(t, \psi)(\omega) = g_{(t,\psi)}(\tilde{I}(\omega)) = g_{(t,\psi)}(\omega) = \omega((t, \psi)) \in \mathbb{R}, \quad \forall \omega \in \mathbb{R}^{\mathbb{R}_+ \times \Psi}.$$

We clearly have that for each $(t, \psi) \in \mathbb{R}^{\mathbb{R}_+ \times \Psi}$, $\tilde{L}(t, \psi)$ is a real-valued random variable since $\{\omega : g_{(t,\psi)}(\tilde{I}(\omega)) < a\} \subseteq \Omega$ is a cylinder set in \mathcal{F} . Moreover, for $A = \{(t_i, \psi_i)\}_{i=1}^n \in \mathbb{A}$, we have that $\tilde{L} \circ A$ given by $\tilde{L} \circ A(\omega) := (\tilde{L}(t_1, \psi_1)(\omega), \dots, \tilde{L}(t_n, \psi_n)(\omega))$ is a random vector because

$$(\tilde{L}(t_1, \psi_1)(\omega), \dots, \tilde{L}(t_n, \psi_n)(\omega)) = (g_{(t_1, \psi_1)} \circ \tilde{I}(\omega), \dots, g_{(t_n, \psi_n)} \circ \tilde{I}(\omega)) = g_A \circ \tilde{I}(\omega) \tag{3.10}$$

and $\{\omega : g_A \circ \tilde{I}(\omega) < a\} \subseteq \Omega$ is also a cylinder set in \mathcal{F} . Then, from (3.9) and (3.10) we have:

$$\mu_A = \mathbb{P} \circ g_A^{-1} = \mathbb{P} \circ (\tilde{L} \circ A)^{-1}, \quad \forall A \in \mathcal{A}. \tag{3.11}$$

Therefore, μ_A is the distribution of the random vector $\tilde{L} \circ A$. Moreover, for any $t \geq 0$ and $\psi_1, \psi_2, \dots, \psi_n$, it follows from our definition of \mathbb{A} that $A = \{(t, \psi_i)\}_{i=1}^n \in \mathbb{A}$, and from (3.5), (3.10) and (3.11) we have for this A that for every $\Gamma \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{P} \left((\tilde{L}(t, \psi_1), \tilde{L}(t, \psi_2), \dots, \tilde{L}(t, \psi_n)) \in \Gamma \right) = \mu_A(\Gamma) = \mu_t \circ \pi_{\psi_1, \psi_2, \dots, \psi_n}^{-1}(\Gamma). \tag{3.12}$$

Now, fix $t \geq 0$. We will show the linearity of the map $\tilde{L}(t, \cdot) : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$.

For any $\psi_1, \psi_2 \in \Psi$, consider the map $\xi : \Psi' \rightarrow \mathbb{R}^3$, given by $f \mapsto (f[\psi_1], f[\psi_2], f[\psi_1 + \psi_2])$. If $\sigma : \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by $(a, b, c) \mapsto a + b - c$, then it is clear that σ is continuous and that $\sigma \circ \xi = 0$. It then follows that for $\Gamma \in \mathcal{B}(\mathbb{R})$, $\mu_t \circ \pi_F^{-1}(\sigma^{-1}(\Gamma)) = 0$ if $0 \notin \Gamma$ and $\mu_t \circ \pi_F^{-1}(\sigma^{-1}(\Gamma)) = 1$ if $0 \in \Gamma$, where $F = (\psi_1, \psi_2, \psi_1 + \psi_2)$. Hence, $\mu_t \circ \pi_F^{-1}$ is supported by the plane $\sigma^{-1}(\{0\}) = \{(a, b, c) : a + b - c = 0\}$ of \mathbb{R}^3 . But then, we have from (3.12) that

$$\mathbb{P} \left((\tilde{L}(t, \psi_1), \tilde{L}(t, \psi_2), \tilde{L}(t, \psi_1 + \psi_2)) \in \sigma^{-1}(\{0\}) \right) = \mu_t \circ \pi_F^{-1}(\sigma^{-1}(\Gamma)) = 0$$

Therefore,

$$\tilde{L}(t, \psi_1) + \tilde{L}(t, \psi_2) = \tilde{L}(t, \psi_1 + \psi_2) \quad \mathbb{P} - \text{a.e.} \quad (3.13)$$

On the other hand, for any $\alpha \in \mathbb{R}$, $\psi \in \Psi$, if we consider $\xi : \Phi \rightarrow \mathbb{R}^2$ given by $f \mapsto (f[\psi], f[\alpha\psi])$ and $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $(p, q) \mapsto \alpha p - q$, by using similar arguments to those used above we can show that

$$\alpha \tilde{L}(t, \psi) = \tilde{L}(t, \alpha\psi) \quad \mathbb{P} - \text{a.e.} \quad (3.14)$$

Hence, (3.13) and (3.14) show that for a fixed $t \geq 0$ the map $\tilde{L}(t, \cdot) : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ is linear.

Now, define $L = \{L_t\}_{t \geq 0}$, $L_t : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ by $L_t(\psi)(\omega) = \tilde{L}(t, \psi)(\omega)$, for all $t \geq 0$, $\psi \in \Psi$ and $\omega \in \Omega$. The linearity of the map $\tilde{L}(t, \cdot) : \Psi \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P})$ for every $t \geq 0$ shows that $L = \{L_t\}_{t \geq 0}$ is a cylindrical stochastic process in Ψ' . Moreover, it follows from (3.12) that $\forall t \geq 0$, $\psi_1, \psi_2, \dots, \psi_n \in \Psi$, $\Gamma \in \mathcal{B}(\mathbb{R}^n)$ we have

$$\mathbb{P}((L_t(\psi_1), L_t(\psi_2), \dots, L_t(\psi_n)) \in \Gamma) = \mu_t \circ \pi_{\psi_1, \psi_2, \dots, \psi_n}^{-1}(\Gamma). \quad (3.15)$$

Now we will show that $L = \{L_t\}_{t \geq 0}$ is a cylindrical Lévy process in Ψ . Fix $\psi_1, \psi_2, \dots, \psi_n \in \Psi$. We have to show that $\{(L_t(\psi_1), L_t(\psi_2), \dots, L_t(\psi_n))\}_{t \geq 0}$ is a \mathbb{R}^n -valued Lévy process.

First, it follows from (3.5), (3.12) and (3.15) that for any $t_1 < t_2 < \dots < t_n$ and any bounded measurable function f on \mathbb{R}^{n^2} , we have

$$\begin{aligned} \mathbb{E}[f((L_{t_1}(\psi_1), L_{t_1}(\psi_2), \dots, L_{t_1}(\psi_n)), \dots, (L_{t_n}(\psi_1), L_{t_n}(\psi_2), \dots, L_{t_n}(\psi_n)))] \\ = \int \dots \int f(w_1, w_1 + w_2, \dots, w_1 + w_2 + \dots + w_n) \\ \times \mu_{t_1}^F(dw_1) \mu_{t_2 - t_1}^F(dw_2 - w_1) \dots \mu_{t_n - t_{n-1}}^F(dw_n - w_{n-1}), \end{aligned} \quad (3.16)$$

where $F = \{(\psi_1, \psi_2, \dots, \psi_n)\} \in \mathbb{F}$. Then, by following similar arguments to those used on the proof of Theorem 2.7.10 in [26] p.36, the independent and stationary increments of $\{(L_t(\psi_1), L_t(\psi_2), \dots, L_t(\psi_n))\}_{t \geq 0}$ can be deduced by fixing $z_1, \dots, z_n \in \mathbb{R}^n$ and setting

$$f(w_1, w_2, \dots, w_n) = \exp \left(i \sum_{j=1}^n \langle z_j, w_j - w_{j-1} \rangle \right), \quad \forall w_1, w_2, \dots, w_n \in \mathbb{R}^n, \text{ with } w_0 = 0.$$

Finally, the fact that the process $\{(L_t(\psi_1), L_t(\psi_2), \dots, L_t(\psi_n))\}_{t \geq 0}$ is stochastically continuous is a consequence of (3.15) and our assumption that $\{\mu_t \circ \pi_{\psi_1, \psi_2, \dots, \psi_n}^{-1}\}_{t \geq 0}$ is a continuous convolution semigroup of probability measures on \mathbb{R}^n (see [1], Proposition 1.4.1). Thus, we have shown that $\{(L_t(\psi_1), L_t(\psi_2), \dots, L_t(\psi_n))\}_{t \geq 0}$ is a \mathbb{R}^n -valued Lévy process for any $\psi_1, \psi_2, \dots, \psi_n \in \Psi$ and consequently $L = \{L_t\}_{t \geq 0}$ is a cylindrical Lévy process in Ψ . \square

We are ready for the main result of this section.

Theorem 3.14. *Let Φ be a barrelled nuclear space. If μ is an infinitely divisible measure on Φ'_β , there exist a Φ'_β -valued, regular, càdlàg Lévy process $L = \{L_t\}_{t \geq 0}$ such that $\mu_{L_1} = \mu$.*

Proof. First, note that as Φ is barrelled then Φ'_β is quasi-complete (see [27], Theorem IV.6.1, p.148). Therefore, it follows from Theorem 3.2 that there exists a unique continuous convolution semigroup $\{\mu_t\}_{t \geq 0}$ in $\mathfrak{M}_R^1(\Phi'_\beta)$ such that $\mu_1 = \mu$.

Now, it is clear that the cylindrical measures determined by the family $\{\mu_t\}_{t \geq 0}$ satisfies that for every finite collection $\phi_1, \phi_2, \dots, \phi_n \in \Phi$, the family $\{\mu_t \circ \pi_{\phi_1, \phi_2, \dots, \phi_n}^{-1}\}_{t \geq 0}$ is a continuous convolution semigroup of probability measures on \mathbb{R}^n . Then, Theorem 3.13 shows the existence of a cylindrical Lévy process $L = \{L_t\}_{t \geq 0}$ in Φ' defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that for every $t \geq 0$, $\phi_1, \phi_2, \dots, \phi_n \in \Phi$ and $\Gamma \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbb{P}((L_t(\phi_1), L_t(\phi_2), \dots, L_t(\phi_n)) \in \Gamma) = \mu_t \circ \pi_{\phi_1, \phi_2, \dots, \phi_n}^{-1}(\Gamma). \quad (3.17)$$

Now, as from Lemma 3.3 the family $\{\mu_t : t \in [0, T]\}$ is uniformly tight for each $T > 0$, it follows from Theorem 2.7 that for every $T > 0$, the family of linear maps $\{L_t : t \in [0, T]\}$ is equicontinuous. But as $L = \{L_t\}_{t \geq 0}$ is a cylindrical Lévy process in Φ' , it follows from Theorem 3.8 that there exists a Φ'_β -valued, regular, càdlàg Lévy process $\tilde{L} = \{\tilde{L}_t\}_{t \geq 0}$ that is a version of $L = \{L_t\}_{t \geq 0}$. Moreover, it follows from (3.17) that for every $t \geq 0$, $\phi_1, \phi_2, \dots, \phi_n \in \Phi$ and $\Gamma \in \mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned} \mu_{\tilde{L}_t} \circ \pi_{\phi_1, \phi_2, \dots, \phi_n}^{-1}(\Gamma) &= \mathbb{P}(\tilde{L}_t \in \pi_{\phi_1, \phi_2, \dots, \phi_n}^{-1}(\Gamma)) \\ &= \mathbb{P}((L_t(\phi_1), L_t(\phi_2), \dots, L_t(\phi_n)) \in \Gamma) = \mu_t \circ \pi_{\phi_1, \phi_2, \dots, \phi_n}^{-1}(\Gamma). \end{aligned}$$

Hence, for every $t \geq 0$, the measures $\mu_{\tilde{L}_t}$ and μ_t coincide on all the cylindrical sets, but as both measures are Radon measures this is enough to conclude that $\mu_{\tilde{L}_t} = \mu_t$. Now, as $\mu_1 = \mu$, we then have that $\mu_{\tilde{L}_1} = \mu$. This finishes the proof. \square

3.4 Wiener Processes in the Dual of a Nuclear Space

In this section we quickly review some properties of Wiener processes in Φ'_β proved by K. Itô [15] and that we will need later for our proof of the Lévy-Itô decomposition.

Definition 3.15. A Φ'_β -valued continuous Lévy process $W = \{W_t\}_{t \geq 0}$ is called a Φ'_β -valued *Wiener process*. A Φ'_β -valued process $G = \{G_t\}_{t \geq 0}$ is called *Gaussian* if for any $n \in \mathbb{N}$ and any $\phi_1, \dots, \phi_n \in \Phi$, $\{(G_t[\phi_1], \dots, G_t[\phi_n]) : t \geq 0\}$ is a Gaussian process in \mathbb{R}^n .

Theorem 3.16 ([15], Theorem 2.7.1). *Let $W = \{W_t\}_{t \geq 0}$ be a Φ'_β -valued Wiener process. Then, W is Gaussian and hence square integrable. Moreover, there exists $\mathbf{m} \in \Phi'_\beta$ and a continuous Hilbertian semi-norm \mathcal{Q} on Φ , called respectively the mean and the covariance functional of W , such that*

$$\mathbb{E}(W_t[\phi]) = t\mathbf{m}[\phi], \quad \forall \phi \in \Phi, t \geq 0. \quad (3.18)$$

$$\mathbb{E}((W_t - t\mathbf{m})[\phi] (W_s - s\mathbf{m})[\varphi]) = (t \wedge s)\mathcal{Q}(\phi, \varphi), \quad \forall \phi, \varphi \in \Phi, s, t \geq 0. \quad (3.19)$$

where in (3.19) $\mathcal{Q}(\cdot, \cdot)$ corresponds to the continuous, symmetric, non-negative bilinear form on $\Phi \times \Phi$ associated to \mathcal{Q} . Furthermore, the characteristic function of W is given by

$$\mathbb{E}\left(e^{iW_t[\phi]}\right) = \exp\left(it\mathbf{m}[\phi] - \frac{t}{2}\mathcal{Q}(\phi)^2\right), \quad \text{for each } t \geq 0, \phi \in \Phi. \quad (3.20)$$

Theorem 3.17. [[15], Theorem 2.7.2] *Given $\mathbf{m} \in \Phi'_\beta$ and a continuous Hilbertian semi-norm \mathcal{Q} on Φ , there exists a Φ'_β -valued Wiener process $W = \{W_t\}_{t \geq 0}$ such that \mathbf{m} and \mathcal{Q} are the mean and covariance functional of W . Moreover, such a process is unique in distribution.*

4 The Lévy-Itô Decomposition.

Assumption 4.1. *We will consider the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, that satisfies the usual conditions, i.e. it is right continuous and \mathcal{F}_0 contains all sets of \mathbb{P} -measure zero.*

We will consider a Φ'_β -valued Lévy process $L = \{L_t\}_{t \geq 0}$ and we assume that:

- (1) $L_t - L_s$ is independent of \mathcal{F}_s for all $0 \leq s < t$.

(2) There exists a countably Hilbertian topology ϑ_L on Φ such that L is a $(\widetilde{\Phi_{\vartheta_L}})'_{\beta}$ -valued càdlàg process.

We denote by $\Omega_L \subseteq \Omega$ a set with $\mathbb{P}(\Omega_L) = 1$ and such that for each $\omega \in \Omega_L$ the map $t \mapsto L_t(\omega)$ is càdlàg in $(\widetilde{\Phi_{\vartheta_L}})'_{\beta}$.

Note that Assumption 4.1(2) implies that L is a regular càdlàg process in Φ'_{β} . It is very important to remark that Assumption 4.1(2) is always satisfied if Φ is a barrelled nuclear space (see Corollary 3.11). The following is a consequence of Assumption 4.1(2):

Lemma 4.2. *For every $\omega \in \Omega_L$ and $T > 0$ there exists a continuous Hilbertian semi-norm $\varrho = \varrho(\omega, T)$ on Φ such that the map $t \mapsto L_t(\omega)$ is càdlàg from $[0, T]$ into the Hilbert space Φ'_{ϱ} .*

Proof. Let $\omega \in \Omega_L$ and $T > 0$. We have for every $t \geq 0$ that $L_t(\omega) \in (\widetilde{\Phi_{\vartheta_L}})'_{\beta} = \mathcal{L}(\widetilde{\Phi_{\vartheta_L}}, \mathbb{R})$. Also, for every fixed $\phi \in \Phi$ the fact that the map $t \mapsto L_t(\omega)$ is càdlàg in $(\widetilde{\Phi_{\vartheta_L}})'_{\beta}$ implies that $\{L_t(\omega)[\phi] : t \in [0, T]\}$ is bounded in \mathbb{R} . Then, because the space $\widetilde{\Phi_{\vartheta_L}}$ is a Fréchet space and hence barrelled, the Banach-Steinhaus theorem (see [20], Theorem 11.9.1, p.400) shows that the set $\{L_t(\omega) : t \in [0, T]\} \subseteq \mathcal{L}(\widetilde{\Phi_{\vartheta_L}}, \mathbb{R})$ is equicontinuous. Therefore, there exists a continuous Hilbertian semi-norm $q = q(\omega, T)$ on $\widetilde{\Phi_{\vartheta_L}}$ (and hence on Φ) such that $\sup_{t \in [0, T]} q'(L_t(\omega)) \leq 1$. By choosing a further continuous Hilbertian semi-norm $\varrho = \varrho(\omega, T)$ on Φ such that $q \leq \varrho$ and $i_{q, \varrho}$ is Hilbert-Schmidt, we obtain that $\sup_{t \in [0, T]} \varrho'(L_t(\omega))^2 \leq \|i_{q, \varrho}\|_{\mathcal{L}_2(\Phi_{\varrho}, \Phi_q)}^2 < \infty$. Then, $L_t(\omega) \in \Phi'_{\varrho}$ for every $t \in [0, T]$. Furthermore, by an application of Parseval's identity, dominated convergence and the fact that for each $\phi \in \Phi$ the map $t \mapsto L_t(\omega)[\phi]$ is càdlàg, it follows that the map $t \mapsto L_t(\omega)$ is càdlàg from $[0, T]$ into the Hilbert space Φ'_{ϱ} ; see the proof of Proposition 3.3 in [12] for the details. \square

4.1 Poisson Random Measures and Poisson Integrals.

In this section we study basic properties of the Poisson integrals defined by a stationary Poisson Point process and its associated Poisson random measure on the dual of a nuclear space (see [14], Sections 1.8 and 1.9, for the basic definitions). For our proof of the Lévy-Itô decomposition we will follow a program that can be thought as an infinite dimensional version of the arguments in [6], where the Poisson integrals will play a central role.

Let $p = \{p_t\}_{t \geq 0}$ be a $\{\mathcal{F}_t\}$ -adapted stationary Poisson point process on $(\Phi'_{\beta}, \mathcal{B}(\Phi'_{\beta}))$. Let N be the Poisson random measure on $[0, \infty) \times \Phi'_{\beta}$ associated to p , i.e.

$$N_p(t, A)(\omega) = \sum_{0 \leq s \leq t} \mathbb{1}_A(p_s(\omega)), \quad \forall \omega \in \Omega, t \geq 0, A \in \mathcal{B}(\Phi'_{\beta}). \quad (4.1)$$

As p is stationary, there exists a Borel measure ν_p on Φ'_{β} such that

$$\mathbb{E}(N_p(t, A)) = t\nu_p(A), \quad \forall t \geq 0, A \in \mathcal{B}(\Phi'_{\beta}). \quad (4.2)$$

We call ν_p the characteristic measure of p .

Let $A \in \mathcal{B}(\Phi'_{\beta})$ with $\nu_p(A) < \infty$. For each $t \geq 0$ the *Poisson integral with respect to N_p* is defined by

$$J_t^{(p)}(A)(\omega) := \int_A f N_p(t, df)(\omega) = \sum_{0 \leq s \leq t} p_s(\omega) \mathbb{1}_A(p_s(\omega)), \quad \forall \omega \in \Omega. \quad (4.3)$$

From now on we assume that $p = \{p_t\}_{t \geq 0}$ is a regular process in $(\Phi'_{\beta}, \mathcal{B}(\Phi'_{\beta}))$. The following result contains the main properties of the Poisson integral process.

Proposition 4.3. *The process $J^{(p)}(A) = \{J_t^{(p)}(A)\}_{t \geq 0}$ is a $\{\mathcal{F}_t\}$ -adapted Φ'_{β} -valued regular càdlàg Lévy process. For every $t \geq 0$ the distribution of $J_t^{(p)}(A)$ is given by*

$$\mathbb{P}\left(\omega : J_t^{(p)}(A)(\omega) \in \Gamma\right) = e^{-t\nu_p(A)} \sum_{k=0}^{\infty} \frac{t^k}{k!} (\nu_p|_A)^{*k}(\Gamma), \quad \forall \Gamma \in \mathcal{B}(\Phi'_{\beta}). \quad (4.4)$$

and its characteristic function is

$$\mathbb{E} \left(\exp \left\{ i J_t^{(p)}(A)[\phi] \right\} \right) = \exp \left\{ t \int_A \left(e^{if[\phi]} - 1 \right) \nu_p(df) \right\}, \quad \forall \phi \in \Phi. \quad (4.5)$$

Moreover, if $\int_A |f[\phi]| \nu_p(df) < \infty$ for each $\phi \in \Phi$, then

$$\mathbb{E} \left(J_t^{(p)}(A)[\phi] \right) = t \int_A f[\phi] \nu_p(df), \quad \forall \phi \in \Phi, \quad (4.6)$$

Furthermore, if $\int_A |f[\phi]|^2 \nu_p(df) < \infty$ for each $\phi \in \Phi$, then

$$\text{Var} \left(J_t^{(p)}(A)[\phi] \right) = t \int_A |f[\phi]|^2 \nu_p(df), \quad \forall \phi \in \Phi. \quad (4.7)$$

Proof. The fact that $J^{(p)}(A)$ is a $\{\mathcal{F}_t\}$ -adapted càdlàg regular process with independent and stationary increments is immediate from the corresponding properties of the processes p , $\{N_p(t, A)\}_{t \geq 0}$ and from (4.3). It is clear from (4.3) that $J_0^{(p)}(A) = 0$ \mathbb{P} -a.e.

The proofs of (4.4), (4.5), (4.6) and (4.7) follows from similar arguments to those used in the proofs of Theorems 2.3.7 and 2.3.9 in [1] where analogous results are proved for the case of Poisson integrals defined by the Poisson random measure of a \mathbb{R}^d -valued Lévy process.

Finally, let $G \in C_b(\Phi'_\beta)$ and let $N > 0$ such that $\sup_{f \in \Phi'} |G(f)| \leq N$. Then, from (4.4) we have:

$$\begin{aligned} \lim_{t \rightarrow 0+} \left| \int_{\Phi'} G(f) \mu_{J_t^{(p)}(A)}(df) - \int_{\Phi'} G(f) \delta_0(df) \right| &\leq \lim_{t \rightarrow 0+} \left| e^{-t\nu_p(A)} \sum_{k=1}^{\infty} \frac{(t\nu_p(A)N)^k}{k!} \right| \\ &= \lim_{t \rightarrow 0+} e^{-t\nu_p(A)} \left(e^{-t\nu_p(A)N} - 1 \right) = 0. \end{aligned}$$

Then, it follows that the map $t \mapsto \mu_{J_t^{(p)}(A)}$ is weakly continuous. Hence, $J^{(p)}(A)$ is a Φ'_β -valued Lévy process. \square

Now, if $\int_A |f[\phi]| \nu_p(df) < \infty$ for each $\phi \in \Phi$, then for each $t \geq 0$ we define the *compensated Poisson integral with respect to N_p* by

$$\tilde{J}_t^{(p)}(A)[\phi] := \int_A f \tilde{N}_p(t, df)[\phi] = \int_A f N_p(t, df)[\phi] - t \int_A f[\phi] \nu_p(df), \quad \forall \phi \in \Phi. \quad (4.8)$$

The process $\tilde{J}^{(p)}(A) = \{\tilde{J}_t^{(p)}(A)\}_{t \geq 0}$ is a Φ'_β -valued, zero-mean, square integrable $\{\mathcal{F}_t\}$ -adapted regular càdlàg Lévy process. In particular, for each $\phi \in \Phi$ the process $\tilde{J}^{(p)}(A)[\phi]$ is a real-valued martingale. Moreover, for each $t \geq 0$ it follows from (4.5) and (4.7) that

$$\mathbb{E} \left(\exp \left\{ i \tilde{J}_t^{(p)}(A)[\phi] \right\} \right) = \exp \left\{ t \int_A \left(e^{if[\phi]} - 1 - if[\phi] \right) \nu_p(df) \right\}, \quad \forall \phi \in \Phi. \quad (4.9)$$

Furthermore, if $\int_A |f[\phi]|^2 \nu_p(df) < \infty$, for each $\phi \in \Phi$, then

$$\mathbb{E} \left(\left| \tilde{J}_t^{(p)}(A)[\phi] \right|^2 \right) = t \int_A |f[\phi]|^2 \nu_p(df), \quad \forall \phi \in \Phi. \quad (4.10)$$

Other important properties of Poisson integrals are summarized in the following result.

Theorem 4.4. *Let $A_1, A_2 \in \mathcal{B}(\Phi'_\beta)$ disjoint sets with $\nu_p(A_1), \nu_p(A_2) < \infty$. Then the processes $J^{(p)}(A_1)$ and $J^{(p)}(A_2)$ are independent. If moreover $\int_{A_i} |f[\phi]| \nu_p(df) < \infty$, for all $\phi \in \Phi$, $i = 1, 2$, then the processes $\tilde{J}^{(p)}(A_1)$ and $\tilde{J}^{(p)}(A_2)$ are independent.*

Proof. Let $\phi_1, \dots, \phi_n \in \Phi$. Then, it follows from (4.3) that the \mathbb{R}^n -valued stochastic processes $(J^{(p)}(A_1)[\phi_1], \dots, J^{(p)}(A_1)[\phi_n])$ and $(J^{(p)}(A_2)[\phi_1], \dots, J^{(p)}(A_2)[\phi_n])$ are compound Poisson processes whose jumps occurs at distinct times for each $\omega \in \Omega$ due to the fact that A_1 and A_2 are disjoint. Then, the same arguments of the proof of Theorem 2.4.6 of [1] p.116 show that the processes $(J^{(p)}(A_1)[\phi_1], \dots, J^{(p)}(A_1)[\phi_n])$ and $(J^{(p)}(A_2)[\phi_1], \dots, J^{(p)}(A_2)[\phi_n])$ are independent. Then, as the processes $J^{(p)}(A_1)$ and $J^{(p)}(A_2)$ are regular it follows from Proposition 2.4 that they are independent.

Now, if the integrability condition $\int_{A_i} |f[\phi]| \nu_p(df) < \infty$, for all $\phi \in \Phi$, $i = 1, 2$, is satisfied, the independence of $\tilde{J}^{(p)}(A_1)$ and $\tilde{J}^{(p)}(A_2)$ follows immediately from the independence of $J^{(p)}(A_1)$ and $J^{(p)}(A_2)$. \square

4.2 The Poisson random measure and Poisson integrals of a Lévy process

For the Lévy process $L = \{L_t\}_{t \geq 0}$, we define by $\Delta L_t := L_t - L_{t-}$ the *jump* of the process L at the time $t \geq 0$. Note that from Assumption 4.1 we have that $\Delta L = \{\Delta L_t\}_{t \geq 0}$ is an $\{\mathcal{F}_t\}$ -adapted Φ'_β -valued regular stochastic process.

We say that a set $A \in \mathcal{B}(\Phi'_\beta \setminus \{0\})$ is *bounded below* if $0 \notin \overline{A}$, where \overline{A} is the closure of A . Then, A is bounded below if and only if A is contained in the complement of a neighborhood of zero. We denote by \mathcal{A} the collection of all the subsets of $\Phi'_\beta \setminus \{0\}$ that are bounded below. Clearly, \mathcal{A} is a ring.

For $A \in \mathcal{B}(\Phi'_\beta \setminus \{0\})$ and $t \geq 0$ define

$$N(t, A)(\omega) = \# \{0 \leq s \leq t : \Delta L_s(\omega) \in A\} = \sum_{0 \leq s \leq t} \mathbb{1}_A(\Delta L_s(\omega)), \quad \text{if } \omega \in \Omega_L$$

and $N(t, A)(\omega) = 0$ if $\omega \in \Omega_L^c$.

From Lemma 4.2, for every $\omega \in \Omega_L$ and $t \geq 0$, there exists a continuous Hilbertian semi-norm $\varrho = \varrho(\omega, t)$ on Φ such that the map $s \mapsto L_s(\omega)$ is càdlàg from $[0, t]$ into the Hilbert space Φ'_ϱ . But as Φ'_ϱ is a complete separable metric space, the above implies that $\Delta L_s(\omega) \neq 0$ for a finite number of $s \in [0, t]$. Hence, $A \mapsto N(t, A)(\omega)$ is a counting measure on $(\Phi'_\beta \setminus \{0\}, \mathcal{B}(\Phi'_\beta \setminus \{0\}))$. Then, $\Delta L = \{\Delta L_t\}_{t \geq 0}$ is a regular stationary Poisson point processes on $(\Phi'_\beta \setminus \{0\}, \mathcal{B}(\Phi'_\beta \setminus \{0\}))$ and $N = \{N(t, A) : t \geq 0, A \in \mathcal{B}(\Phi'_\beta \setminus \{0\})\}$ is the *Poisson random measure* associated to ΔL with respect to the ring \mathcal{A} . Let ν be the characteristic measure of ΔL , i.e. the Borel measure on Φ'_β defined by $\nu(\{0\}) = 0$ and that satisfies:

$$\mathbb{E}(N(t, \Gamma)) = t\nu(\Gamma), \quad \forall t \geq 0, \Gamma \in \mathcal{B}(\Phi'_\beta \setminus \{0\}). \quad (4.11)$$

Clearly, $\nu(A) < \infty$ for every $A \in \mathcal{A}$.

Definition 4.5. Let μ be a Borel measure on Φ'_β . We will say that μ is a θ -regular measure on Φ'_β if there exists a weaker countably Hilbertian topology θ on Φ such that μ is concentrated on Φ'_θ , i.e. $\mu(\Phi \setminus \Phi'_\theta) = 0$.

Lemma 4.6. The measure ν is θ_L -regular (where θ_L is as in Assumption 4.1). Moreover, for every $A \in \mathcal{B}(\Phi'_\beta)$ such that $\nu(A) < \infty$ (in particular if $A \in \mathcal{A}$), $\nu|_A$ is θ_L -regular and $\nu|_A \in \mathfrak{M}_R^b(\Phi'_\beta)$.

Proof. First, note that from Assumption 4.1(2) we have that $\Delta L_t \in \Phi'_{\theta_L} \forall t \geq 0$ \mathbb{P} -a.e. and hence from (4.11) we have that $\nu(\Phi'_\beta \setminus \Phi'_{\theta_L}) = 0$ and hence ν is θ_L -regular.

Now, let $A \in \mathcal{B}(\Phi'_\beta)$ such that $\nu(A) < \infty$. Because the measure ν is θ_L -regular then the measure $\nu|_A$ is also. If we consider the canonical Φ'_β -valued random variable $X_{\nu, A}$ whose probability distribution is $\frac{\nu|_A(\cdot)}{\nu|_A(\Phi'_{\theta_L})}$, we then have that $\mathbb{P}(X_{\nu, A} \in \Phi'_{\theta_L}) = 1$ and hence $X_{\nu, A}$ is a regular random variable. Therefore, Theorem 2.2 shows that the probability distribution of $X_{\nu, A}$ is a Radon measure on Φ'_β . Then, $\nu|_A \in \mathfrak{M}_R^b(\Phi'_\beta)$. \square

For every $A \in \mathcal{B}(\Phi'_\beta)$ such that $\nu(A) < \infty$, we will denote by $J(A)$ the Poisson integral with respect to N and if $\int_A |f[\phi]|^2 \nu(df) < \infty$, for each $\phi \in \Phi$, we denote by $\tilde{J}(A)$ the compensated Poisson integral with respect to N .

Theorem 4.7. *Let $A \in \mathcal{B}(\Phi'_\beta)$ with $\nu(A) < \infty$. Then, $L - J(A) = \{L_t - J_t(A)\}_{t \geq 0}$ is a Φ'_β -valued Lévy process. Moreover, the processes $L - J(A)$ and $J(A)$ are independent.*

Proof. First, the same arguments to those used in Theorem 2.4.8 of [1] for the case of \mathbb{R}^n -valued Lévy processes shows that $L - J(A)$ is a Φ'_β -valued Lévy process. To prove the independence of $L - J(A)$ and $J(A)$, let $\phi_1, \dots, \phi_n \in \Phi$. As $((L - J(A))[\phi_1], \dots, (L - J(A))[\phi_n])$ and $(J(A)[\phi_1], \dots, J(A)[\phi_n])$ are \mathbb{R}^n -valued Lévy processes that have their jumps at distinct times for each $\omega \in \Omega$, the same arguments of the proof of Lemma 7.9 and Theorem 7.12 of [19] p.468-71 show that the processes $((L - J(A))[\phi_1], \dots, (L - J(A))[\phi_n])$ and $(J(A)[\phi_1], \dots, J(A)[\phi_n])$ are independent. Then, the independence of $L - J(A)$ and $J(A)$ follows from Proposition 2.4 as both $L - J(A)$ and $J(A)$ are regular processes. \square

4.3 Lévy Measures on the Dual of a Nuclear Space

Lévy measures play an important role on the study of Lévy processes and infinitely divisible measures. In this section we introduce our definition of Lévy measure and derive some of its basic properties.

Definition 4.8. A Borel measure λ on Φ'_β is a Lévy measure if

- (1) $\lambda(\{0\}) = 0$,
- (2) for each neighborhood of zero $U \subseteq \Phi'_\beta$, $\lambda|_{U^c} \in \mathfrak{M}_R^b(\Phi'_\beta)$,
- (3) there exists a continuous Hilbertian semi-norm ρ on Φ such that

$$\int_{B_{\rho'}(1)} \rho'(f)^2 \lambda(df) < \infty, \quad \text{and} \quad \lambda|_{B_{\rho'}(1)^c} \in \mathfrak{M}_R^b(\Phi'_\beta), \quad (4.12)$$

where we recall that $B_{\rho'}(1) := \{f \in \Phi' : \rho'(f) \leq 1\} = B_\rho(1)^0$.

Note that (4.12) implies that

$$\int_{\Phi'} (\rho'(f)^2 \wedge 1) \lambda(df) < \infty, \quad (4.13)$$

which resembles the property that characterizes Lévy measures on Hilbert spaces (see [21]).

Remark 4.9. *If Φ is a complete barrelled nuclear space, our definition of Lévy measures on Φ'_β coincides with the characterization of Lévy measures for complete Badrikian spaces given in [8].*

Proposition 4.10. *Every Lévy measure on Φ'_β is σ -finite.*

Proof. Let λ be a Lévy measure on Φ'_β and let ρ as in Definition 4.8(3). From (4.13) and standard arguments we have that $\lambda(B_{\rho'}(\epsilon)^c) < \infty \forall 0 < \epsilon \leq 1$. But the above together with $\lambda(\{0\}) = 0$ imply that λ is σ -finite. \square

Proposition 4.11. *Every Lévy measure on Φ'_β is a Radon measure.*

Proof. Let λ be a Lévy measure on Φ'_β and let ρ as in Definition 4.8(3). Because $\lambda|_{B_{\rho'}(1)^c} \in \mathfrak{M}_R^b(\Phi'_\beta)$, it is enough to show that $\lambda|_{B_{\rho'}(1)} \in \mathfrak{M}_R^b(\Phi'_\beta)$.

To show this, let $q : \Phi \rightarrow \mathbb{R}$ defined by

$$q(\phi)^2 = \int_{B_{\rho'}(1)} |f[\phi]|^2 \lambda(df), \quad \forall \phi \in \Phi.$$

It is clear that q is a Hilbertian semi-norm on Φ . Moreover, because $q(\phi)^2 \leq C\rho(\phi)^2$ for all $\phi \in \Phi$, where $C = \int_{B_{\rho'}(1)} \rho'(f)^2 \lambda(df) < \infty$, then q is continuous on Φ .

Now, note that for every $\phi \in \Phi$ we have

$$1 - \operatorname{Re} \widehat{\lambda|_{B_{\rho'}(1)}}(\phi) = \int_{B_{\rho'}(1)} (1 - \cos f[\phi]) \lambda(df) \leq \frac{1}{2} \int_{B_{\rho'}(1)} f[\phi]^2 \lambda(df) = \frac{1}{2} q(\phi)^2.$$

Then, it follows that $\widehat{\lambda|_{B_{\rho'}(1)}}$ is continuous on Φ . Finally, by Minlos' theorem (see [7], Theorem III.1.3, p.88) this shows that $\lambda|_{B_{\rho'}(1)}$ is a Radon measure on Φ'_β . Therefore, $\lambda \in \mathfrak{M}_R^b(\Phi'_\beta)$. \square

Corollary 4.12. *If Φ is a barrelled nuclear space, every Lévy measure on Φ'_β is θ -regular.*

Proof. Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of real numbers satisfying $0 < \epsilon_n \leq 1$ and such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Because λ is a Radon measure on Φ (Proposition 4.11), there exists an increasing (under set inclusion) sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of Φ'_β such that $\lambda(K_n^c) < \epsilon_n$.

Now, because Φ is barrelled, similar arguments to those used in the second paragraph in the proof of Theorem 2.7 shows that for every $n \in \mathbb{N}$ there exists a continuous Hilbertian semi-norm p_n on Φ such that $K \subseteq B_{p'_n}(1)$. We can and will assume without loss of generality that the sequence $\{p_n\}_{n \in \mathbb{N}}$ is increasing and hence we have $B_{p'_n}(n) \subseteq B_{p'_m}(m)$ for $n \leq m$.

Let θ be the weaker countably Hilbertian topology on Φ generated by the semi-norms $\{p_n\}_{n \in \mathbb{N}}$. Then,

$$\Phi'_\theta = \bigcup_{n \in \mathbb{N}} \Phi'_{p_n} = \bigcup_{n \in \mathbb{N}} B_{p'_n}(n).$$

But as $\lambda(B_{p'_n}(n)^c) \leq \lambda(B_{p'_n}(1)^c) \leq \lambda(K^c) < \epsilon_n$ for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \epsilon_n = 0$, we then have that

$$\lambda(\Phi \setminus \Phi'_\theta) = \lim_{n \rightarrow \infty} \lambda(B_{p'_n}(1)^c) = 0.$$

Therefore, λ is a θ -regular measure on Φ'_β . \square

4.4 The Lévy Measure of a Lévy process

We proceed to show that the measure ν associated to the Poisson measure N of the Lévy process L is a Lévy measure on Φ'_β . We start by recalling the concept of Poisson measures that will be of great importance for our arguments.

Let $\mu \in \mathfrak{M}_R^b(\Phi'_\beta)$. The measure $e(\mu) \in \mathfrak{M}_R^1(\Phi'_\beta)$ defined by

$$e(\mu)(\Gamma) = e^{-\mu(\Phi'_\beta)} \sum_{k=0}^{\infty} \frac{1}{k!} \mu^{*k}(\Gamma), \quad \forall \Gamma \in \mathcal{B}(\Phi'_\beta),$$

is called a *Poisson measure*. We call μ the *Poisson exponent* of $e(\mu)$. It is clear that $e(\mu)$ is infinitely divisible and that

$$\widehat{e(\mu)}(\phi) = \exp[-(\widehat{\mu}(0) - \widehat{\mu}(\phi))], \quad \forall \phi \in \Phi. \quad (4.14)$$

Very important for our forthcoming arguments will be the fact that for $\mu \in \mathfrak{M}_R^b(\Phi'_\beta)$, $|\widehat{e(\mu)}(\phi)|^2$ is the characteristic function of a measure belonging to $\mathfrak{M}_R^b(\Phi'_\beta)$. Indeed, it is the characteristic function of the measure $e(\mu + \overline{\mu}) = e(\mu) * e(\overline{\mu})$, where $\overline{\mu} \in \mathfrak{M}_R^b(\Phi'_\beta)$ is defined by $\overline{\mu}(\Gamma) = \mu(-\Gamma)$ for all $\Gamma \in \mathcal{B}(\Phi'_\beta)$.

Now to show that ν is a Lévy measure we will need two preliminary results. The following is a mild generalization of a result due to Fernique for the characteristic function of infinitely divisible measures on \mathscr{D}' . Its proof easily extends to our case so we omit it and refer the reader to [10], Corollaire 2.

Lemma 4.13. *Let μ be an infinitely divisible measure on Φ'_β . Then, for every continuous Hilbertian seminorm p on Φ and every $\epsilon \in]0, \frac{1}{4}]$ such that:*

$$\forall \phi \in \Phi, \quad p(\phi) \leq 1 \quad \Rightarrow \quad |1 - \widehat{\mu}(\phi)| < \epsilon,$$

we have that

$$\forall \phi \in \Phi, \forall n \in \mathbb{N}, \quad n \cdot (1 - \operatorname{Re} \widehat{\mu}^{1/n}(\phi)) \leq 8\epsilon(1 + p(\phi)^2).$$

Another result that will be of great importance for our forthcoming arguments is the following version of Minlos' lemma due to Fernique. With some modifications, its proof can be carried out as the proof of Lemme 2 in [9] for bounded measures on \mathcal{D}' .

Lemma 4.14 (Minlos' lemma). *Let $\mu \in \mathfrak{M}_R^b(\Phi'_\beta)$. Suppose that there exists $\epsilon > 0$ and a continuous Hilbertian seminorm p on Φ such that*

$$1 - \operatorname{Re} \widehat{\mu}(\phi) \leq \epsilon(1 + p(\phi)^2), \quad \forall \phi \in \Phi.$$

If q is any continuous Hilbertian seminorm on Φ , $p \leq q$ and such that $i_{p,q}$ is Hilbert-Schmidt, then we have that

$$\int_{\Phi'} (q'(f))^2 \wedge 1 \mu(df) \leq \epsilon \left(1 + \|i_{p,q}\|_{\mathcal{L}_2(\Phi_q, \Phi_p)}^2\right) < \infty.$$

We are ready for the main result of this section:

Theorem 4.15. *The measure ν of the Φ'_β -valued Lévy process L is a Lévy measure on Φ'_β .*

Proof. By definition $\nu(\{0\}) = 0$. Now, because for every neighborhood of zero $U \subseteq \Phi'_\beta$, we have that $U^c \in \mathcal{A}$, then $\nu|_{U^c} \in \mathfrak{M}_R^b(\Phi'_\beta)$ (Lemma 4.6). Therefore, it only remains to show that there exists a continuous Hilbertian semi-norm ρ on Φ such that ν satisfies (4.13) with λ replaced by ν . This is because (4.13) implies that $\nu(B_{\rho'}(1)^c) < \infty$ and hence from Lemma 4.6 we obtain that $\nu|_{B_{\rho'}(1)^c} \in \mathfrak{M}_R^b(\Phi'_\beta)$. For our proof, we will benefit from some arguments of the proof of Lemma 2.1 in [8].

Let \mathfrak{B} be a local base of closed neighborhoods of zero for Φ'_β and let $\mathcal{A}_\mathfrak{B} = \{V^c : V \in \mathfrak{B}\}$. Because Φ'_β is Hausdorff, it follows that $\Phi'_\beta \setminus \{0\} = \bigcup_{A \in \mathcal{A}_\mathfrak{B}} A$.

For each $A \in \mathcal{A}_\mathfrak{B}$, let $\nu_A := \nu|_A$. As each $A \in \mathcal{A}_\mathfrak{B}$ satisfies $A \in \mathcal{A}$, we have that $\nu_A \in \mathfrak{M}_R^b(\Phi'_\beta)$ for all $A \in \mathcal{A}_\mathfrak{B}$ (Lemma 4.6). Now consider on $\mathcal{A}_\mathfrak{B}$ the order relationship given by the inclusion of sets. Then, $\{\nu_A\}_{A \in \mathcal{A}_\mathfrak{B}}$ is an increasing net (setwise) in $\mathfrak{M}_R^b(\Phi'_\beta)$. Moreover, because $\mathcal{A}_\mathfrak{B}$ is an increasing net of open subsets that satisfies $\Phi'_\beta \setminus \{0\} = \bigcup_{A \in \mathcal{A}_\mathfrak{B}} A$, and ν can be reduced to be a Borel measure on the (separable and metrizable) subspace Φ'_{θ_L} of Φ'_β (this from Assumption 4.1(2) and (4.11)), it follows that $\nu = \sup_{A \in \mathcal{A}_\mathfrak{B}} \nu_A$ (setwise) (see [4], Propositions 7.2.2 and 7.2.5).

On the other hand, note that from Theorem 4.7, for each $A \in \mathcal{A}_\mathfrak{B}$, the processes $L - J(A)$ and $J(A)$ are independent. Therefore, we have

$$\widehat{\mu_{L_t}}(\phi) = \widehat{\mu_{L_t - J_t(A)}}(\phi) \cdot \widehat{\mu_{J_t(A)}}(\phi), \quad \forall A \in \mathcal{A}_\mathfrak{B}, t \geq 0, \phi \in \Phi. \quad (4.15)$$

Now, for fixed $A \in \mathcal{A}_\mathfrak{B}$, $t \geq 0$, $\phi \in \Phi$, because $|\widehat{\mu_{L_t - J_t(A)}}(\phi)| \leq 1$ it follows from (4.15) that $|\widehat{\mu_{L_t}}(\phi)|^2 \leq |\widehat{\mu_{J_t(A)}}(\phi)|^2 \leq 1$. Therefore, we have that

$$1 - |\widehat{\mu_{J_t(A)}}(\phi)|^2 \leq 1 - |\widehat{\mu_{L_t}}(\phi)|^2, \quad \forall A \in \mathcal{A}_\mathfrak{B}, t \geq 0, \phi \in \Phi. \quad (4.16)$$

On the other hand, note that if we take $t = 1$ in (4.4) then we have $\mu_{J_t(A)} = e(\nu_A)$, for all $A \in \mathcal{A}$. Therefore, it follows from (4.16) that

$$1 - |\widehat{e(\nu_A)}(\phi)|^2 \leq 1 - |\widehat{\mu_{L_t}}(\phi)|^2, \quad \forall A \in \mathcal{A}_\mathfrak{B}, \phi \in \Phi. \quad (4.17)$$

Now, because L_1 is a regular random variable, it follows from Theorem 2.2 that the map $\phi \mapsto L_t[\phi]$ from Φ into $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. But this in turn implies that $\widehat{\mu_{L_1}}$ and hence $|\widehat{\mu_{L_1}}|^2$ is continuous at zero. Therefore, there exists a continuous Hilbertian semi-norm p on Φ such that

$$\forall \phi \in \Phi, \quad p(\phi) \leq 1 \quad \Rightarrow \quad 1 - |\widehat{\mu_{L_1}}(\phi)|^2 < \frac{1}{4}, \quad \forall \phi \in \Phi. \quad (4.18)$$

Hence, it follows from (4.17) and (4.18) that

$$\forall A \in \mathcal{A}_{\mathfrak{B}}, \phi \in \Phi, \quad p(\phi) \leq 1 \quad \Rightarrow \quad 1 - \left| \widehat{e(\nu_A)}(\phi) \right|^2 < \frac{1}{4}. \quad (4.19)$$

Now, let $\phi \in \Phi$. For every $A \in \mathcal{A}_{\mathfrak{B}}$ and every $n \in \mathbb{N}$, from (4.14) for the measure ν_A , we have

$$-\log \left| \widehat{e(\nu_A)}(\phi) \right|^{2/n} = \frac{2}{n} \int_{\Phi'} (1 - \cos f[x]) \nu_A(df) \leq \frac{4}{n} \nu_A(\Phi'_\beta) < \infty.$$

So for fixed $A \in \mathcal{A}_{\mathfrak{B}}$, by choosing $n \in \mathbb{N}$ sufficiently large such that $\nu_A(\Phi'_\beta) \leq \frac{n}{4}$, and by using the elementary inequality $\frac{t}{4} \leq 1 - e^{-t}$ that is valid for $t \in [0, 1]$, by taking $t = -\log \left| \widehat{e(\nu_A)}(\phi) \right|^{2/n}$ we obtain that

$$\begin{aligned} 1 - \operatorname{Re} \widehat{\nu_A}(\phi) &= \int_{\Phi'} (1 - \cos f[x]) \nu_A(df) \\ &= -\frac{n}{2} \log \left| \widehat{e(\nu_A)}(\phi) \right|^{2/n} \leq 2n \cdot \left(1 - \left| \widehat{e(\nu_A)}(\phi) \right|^{2/n} \right). \end{aligned} \quad (4.20)$$

On the other hand, from (4.19) and Lemma 4.13 (with $\epsilon = \frac{1}{4}$) we have that

$$n \cdot \left(1 - \left| \widehat{e(\nu_A)}(\phi) \right|^{2/n} \right) \leq 2(1 + p(\phi)^2). \quad (4.21)$$

Then, (4.20) and (4.21) shows that

$$1 - \operatorname{Re} \widehat{\nu_A}(\phi) < 4(1 + p(\phi)^2), \quad \forall A \in \mathcal{A}_{\mathfrak{B}}, \phi \in \Phi. \quad (4.22)$$

But from Lemma 4.14, if ρ is any continuous Hilbertian seminorm on Φ , $p \leq \rho$, such that $i_{p,\rho}$ is Hilbert-Schmidt, then above implies that

$$\int_{\Phi'} (\rho'(f)^2 \wedge 1) \nu(df) = \sup_{A \in \mathcal{A}_{\mathfrak{B}}} \int_{\Phi'} (\rho'(f)^2 \wedge 1) \nu_A(df) \leq 4 \left(1 + \|i_{p,\rho}\|_{\mathcal{L}_2(\Phi_\rho, \Phi_p)}^2 \right) < \infty.$$

Hence, ν is a Lévy measure. \square

Definition 4.16. From now on, the measure ν of the Lévy process L will be called the *Lévy measure* of L .

Theorem 4.17. If ν is the Lévy measure of the Lévy process L , then ν is a θ_L -regular (with θ_L as in Assumption 4.1) σ -finite Radon measure on Φ with $\nu(\{0\}) = 0$ and such that there exists an increasing net (setwise) $\{\nu_A\}_{A \in I} \subseteq \mathfrak{M}_R^b(\Phi'_\beta)$ such that:

- (1) $\nu = \sup_{A \in I} \nu_A$ (setwise),
- (2) the family of Poisson measures $\{e(\nu_A)\}_{A \in I}$ is shift tight.

Proof. By definition $\nu(\{0\}) = 0$. Moreover, from Lemma 4.6, Propositions 4.10 and 4.11, and Theorem 4.15, ν is a Radon measure on Φ'_β .

Let \mathfrak{B} be a local base of closed neighborhoods of zero for Φ'_β and let $I = \mathcal{A}_{\mathfrak{B}} := \{V^c : V \in \mathfrak{B}\}$. If we define $\nu_A := \nu|_A$ for each $A \in I$, then it was shown on the proof of Theorem 4.15 that the family $\{\nu_A\}_{A \in I}$ is an increasing net (setwise) in $\mathfrak{M}_R^b(\Phi'_\beta)$ satisfying $\nu = \sup_{A \in I} \nu_A$ (setwise) and such that the characteristic functions of the family of Poisson measures $\{e(\nu_A)\}_{i \in I}$ satisfies (4.17). But as $|\widehat{\mu_{L_1}}|^2$ is continuous at zero on Φ , we then have that the family of characteristic functions $\left\{ \left| \widehat{e(\nu_A)}(\phi) \right|^2 : A \in I \right\}$ is equicontinuous at zero on Φ .

Moreover, because Φ is nuclear, and for each $A \in I$, $\left| \widehat{e(\nu_A)}(\phi) \right|^2$ is the characteristic function of $e(\nu_A + \overline{\nu_A}) = e(\nu_A) * e(\overline{\nu_A})$, then the equicontinuity of the family $\left\{ \left| \widehat{e(\nu_A)}(\phi) \right|^2 : A \in I \right\}$ implies that the family $\{e(\nu_A) * e(\overline{\nu_A})\}_{A \in I}$ is uniformly tight (see [7], Lemma III.2.3, p.103-4). But this last in turn implies that the family $\{e(\nu_A)\}_{A \in I}$ is shift tight (see [13], Theorem 2.2.7, p.41, the arguments there for probability measures on Banach spaces can be modified to hold also in our context). \square

Remark 4.18. *It is a consequence of Theorem 4.17 that the Lévy measure of a Lévy process in Φ'_β is a Lévy measure on the general sense for the context of locally convex spaces (see [8], [31]). Later, in Theorem 4.24 we will show that every θ -regular Lévy measure on Φ'_β , and hence every Lévy measure on Φ'_β when Φ is a barrelled nuclear space, is the Lévy measure of a Φ'_β -valued Lévy processes and hence also satisfies the definition of Lévy measure on the general sense for the context of locally convex spaces.*

4.5 The Lévy-Itô Decomposition.

Our main objective of this section is to prove Theorem 4.23, which is the Lévy-Itô decomposition. We will need the following properties of the space of martingales taking values in the Hilbert spaces Φ'_q .

For a continuous Hilbertian semi-norm q on Φ we denote by $\mathcal{M}^2(\Phi'_q)$ and $\mathcal{M}_T^2(\Phi'_q)$ the linear spaces of (equivalent classes of) Φ'_q -valued zero-mean, square integrable, càdlàg, $\{\mathcal{F}_t\}$ -adapted martingales defined respectively on $[0, \infty)$ and on $[0, T]$ (with $T > 0$).

The space $\mathcal{M}_T^2(\Phi'_q)$, is a Banach space equipped with the norm $\|\cdot\|_{\mathcal{M}_T^2(\Phi'_q)}$ defined by

$$\|M\|_{\mathcal{M}_T^2(\Phi'_q)} = \left(\mathbb{E} \sup_{t \in [0, T]} q'(M_t)^2 \right)^{1/2}, \quad \forall M \in \mathcal{M}_T^2(\Phi'_q).$$

For every $K \in \mathbb{N}$, there exists a canonical inclusion j_K of the space $\mathcal{M}^2(\Phi'_q)$ into the space $\mathcal{M}_K^2(\Phi'_q)$. Therefore, we can equip $\mathcal{M}^2(\Phi'_q)$ with the projective limit topology determined by the projective system $\{(\mathcal{M}_K^2(\Phi'_q), j_K) : K \in \mathbb{N}\}$. Then, equipped with this topology, $\mathcal{M}^2(\Phi'_q)$ is a Fréchet space and a family of semi-norms generating its topology is $\{\|j_K(\cdot)\|_{\mathcal{M}_K^2(\Phi'_q)}\}_{K \in \mathbb{N}}$. In particular, convergence in $\mathcal{M}^2(\Phi'_q)$ is then equivalent to convergence in the space $L^2(\Omega, \mathcal{F}, \mathbb{P}; \Phi'_q)$ uniformly on compact intervals of $[0, \infty)$.

Now we start with our preparations for the proof of Theorem 4.23. Let ν be the Lévy measure of L . According to Definition 4.8 and Theorem 4.15, there exists a continuous Hilbertian semi-norm ρ on Φ such that

$$\int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty, \quad \text{and} \quad \nu|_{B_{\rho'}(1)^c} \in \mathfrak{M}^b(\Phi'_\beta), \quad (4.23)$$

where $B_{\rho'}(1) := B_\rho(1)^0 = \{f \in \Phi'_\beta : \rho'(f) \leq 1\}$. As $B_\rho(1)$ is a convex, balanced, neighborhood of zero, then its polar $B_{\rho'}(1)$ is a bounded, closed, convex, balanced subset of Φ'_β .

Theorem 4.19. *There exists a Φ'_β -valued zero-mean, square integrable, càdlàg Lévy process $M = \{M_t\}_{t \geq 0}$ such that for all $t \geq 0$, it has characteristic function given by*

$$\mathbb{E} \left(e^{iM_t[\phi]} \right) = \exp \left\{ t \int_{B_{\rho'}(1)} \left(e^{if[\phi]} - 1 - if[\phi] \right) \nu(df) \right\}, \quad \forall \phi \in \Phi, \quad (4.24)$$

and second moments given by

$$\mathbb{E} \left(|M_t[\phi]|^2 \right) = t \int_{B_{\rho'}(1)} |f[\phi]|^2 \nu(df), \quad \forall \phi \in \Phi. \quad (4.25)$$

Moreover, there exists a continuous Hilbertian semi-norm q on Φ , $\rho \leq q$, such that $i_{\rho, q}$ is Hilbert-Schmidt and for which M is a Φ'_q -valued zero-mean, square integrable, càdlàg Lévy process with second moment given by

$$\mathbb{E} \left(q'(M_t)^2 \right) = \int_{B_{\rho'}(1)} q'(f)^2 \nu(df), \quad \forall t \geq 0. \quad (4.26)$$

Proof. Let \mathfrak{B} be a local base of closed neighborhoods of zero for Φ'_β . Let $\mathcal{A}_{\rho'}$ denotes the collection of all sets of the form $V \cap B_{\rho'}(1)$, where $V^c \in \mathfrak{B}$. It is clear that $\mathcal{A}_{\rho'} \subseteq \mathcal{A}$ (see Section 4.2). Moreover, as $\Phi'_\beta \setminus \{0\} = \bigcup_{V \in \mathfrak{B}} V^c$ (this follows because Φ'_β is Hausdorff) then we have $B_{\rho'}(1) \setminus \{0\} = \bigcup_{A \in \mathcal{A}_{\rho'}} A$.

Fix an arbitrary $A \in \mathcal{A}_{\rho'}$. It follows from (4.23) that

$$\int_A |f[\phi]|^2 \nu(df) \leq \rho(\phi)^2 \int_A \rho'(f)^2 \nu(df) \leq \rho(\phi)^2 \int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty, \quad \forall \phi \in \Phi. \quad (4.27)$$

Therefore, the compensated Poisson integral $\tilde{J}(A)$ is a Φ'_β -valued zero-mean, square integrable, càdlàg, regular Lévy process with characteristic function given by (4.9) and second moments given by (4.10) (with $\tilde{J}^{(p)}(A)$ replaced by $\tilde{J}_t(A)$ and ν_p by ν). Moreover, for each $\phi \in \Phi$ the process $\tilde{J}_t(A)[\phi]$ is a real-valued $\{\mathcal{F}_t\}$ -adapted martingale. From Doob's inequality, (4.10) and (4.27), for every $T > 0$ we have

$$\mathbb{E} \left(\sup_{t \in [0, T]} \left| \tilde{J}_t(A)[\phi] \right|^2 \right) \leq 4T \mathbb{E} \left(\left| \tilde{J}_T(A)[\phi] \right|^2 \right) \leq C(T) \rho(\phi)^2, \quad \forall \phi \in \Phi,$$

where $C(T) = 4T \int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty$. Then, from Theorem 3.12, there exists a continuous Hilbertian semi-norm q on Φ , $\rho \leq q$, such that $i_{\rho, q}$ is Hilbert-Schmidt and for which $\tilde{J}(A)$ possesses a version that is a càdlàg, zero-mean, square integrable, Lévy process in Φ'_q . We denote this version again by $\tilde{J}(A)$. Let $\{\phi_j^q\}_{j \in \mathbb{N}} \subseteq \Phi$ be a complete orthonormal system of Φ_q . Then, from Fubini's theorem, Parseval's identity and (4.10), for every $t \geq 0$ we have

$$\mathbb{E} \left(q'(\tilde{J}_t(A))^2 \right) = \sum_{j=1}^{\infty} \mathbb{E} \left(\left| \tilde{J}_t(A)[\phi_j^q] \right|^2 \right) = t \sum_{j=1}^{\infty} \int_A |f[\phi_j^q]|^2 \nu(df) = t \int_A q'(f)^2 \nu(df). \quad (4.28)$$

Now, consider on $\mathcal{A}_{\rho'}$ the order induced by the inclusion of sets. Our next objective is to show that for every $T > 0$ the net $\{\{\tilde{J}_t(A)\}_{t \in [0, T]} : A \in \mathcal{A}_{\rho'}\}$ converges in the space $\mathcal{M}_T^2(\Phi'_q)$. To do this, we will show that for a fixed $T > 0$, $\{\{\tilde{J}_t(A)\}_{t \in [0, T]} : A \in \mathcal{A}_{\rho'}\}$ is a Cauchy net in $\mathcal{M}_T^2(\Phi'_q)$, then convergence follows by completeness of this space.

Fix an arbitrary $T > 0$. First observe that if $A_1, A_2 \in \mathcal{A}_{\rho'}$, $A_1 \subseteq A_2$, then from Doob's inequality, the definition of compensated Poisson integral and (4.28) we have

$$\mathbb{E} \left(\sup_{t \in [0, T]} q'(\tilde{J}_t(A_1) - \tilde{J}_t(A_2))^2 \right) \leq 4\mathbb{E} \left(q'(\tilde{J}_T(A_2 \setminus A_1))^2 \right) = 4T \int_{A_2 \setminus A_1} q'(f)^2 \nu(df). \quad (4.29)$$

Therefore, if we can show that

$$\lim_{A \in \mathcal{A}_{\rho'}} \int_A q'(f)^2 \nu(df) = \int_{B_{\rho'}(1)} q'(f)^2 \nu(df) < \infty, \quad (4.30)$$

then (4.29) and (4.30) would show that $\{\tilde{J}^A\}_{A \in \mathcal{A}_{\rho'}}$ is a Cauchy net on $\mathcal{M}_T^2(\Phi'_q)$.

To prove (4.30), note that as ν is a Borel measure on $B_{\rho'}(1)$, and $B_{\rho'}(1)$ is a Suslin set (it is the image under the continuous map i'_ρ of the unit ball of the separable Hilbert space Φ'_ρ), then ν is a Radon measure on $B_{\rho'}(1)$ ([4], Vol II, Theorem 7.4.3, p.85). Moreover, as $B_{\rho'}(1) \setminus \{0\} = \bigcup_{A \in \mathcal{A}_{\rho'}} A$ and because ν is a Radon probability measure on $B_{\rho'}(1)$ such that $\nu(\{0\}) = 0$, we have that $\nu(B_{\rho'}(1)) = \lim_{A \in \mathcal{A}_{\rho'}} \nu(A)$ (see [4], Vol. II, Propositions 7.2.2 and 7.2.5, p.74-5). Therefore, from all the above we have

$$\begin{aligned} \lim_{A \in \mathcal{A}_{\rho'}} \left| \int_{B_{\rho'}(1)} q'(f)^2 \nu(df) - \int_A q'(f)^2 \nu(df) \right| &\leq \lim_{A \in \mathcal{A}_{\rho'}} \int_{B_{\rho'}(1) \setminus A} q'(f)^2 \nu(df) \\ &\leq \sup_{f \in B_{\rho'}(1)} q'(f)^2 \lim_{A \in \mathcal{A}_{\rho'}} \mu(B_{\rho'}(1) \setminus A) = 0, \end{aligned}$$

and hence (4.30) is valid.

Thus, $\{\{\tilde{J}_t(A)\}_{t \in [0, T]} : A \in \mathcal{A}_{\rho'}\}$ is a Cauchy net on $\mathcal{M}_T^2(\Phi'_q)$ for every $T > 0$. This in turn implies that $\{\tilde{J}^A : A \in \mathcal{A}_{\rho'}\}$ converges in $\mathcal{M}^2(\Phi'_q)$. Therefore, there exists some $M = \{M_t\}_{t \geq 0}$ that is a Φ'_q -valued zero-mean, square integrable, càdlàg martingale and such that the net $\{\tilde{J}(A) : A \in \mathcal{A}_{\rho'}\}$ converges to M in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \Phi'_q)$ uniformly on compact intervals of $[0, \infty)$. This uniform convergence, (4.28) and (4.30) implies that M satisfies (4.26). Moreover, viewing M as a Φ'_β -valued processes it is also a Φ'_β -valued, zero-mean, square integrable, càdlàg martingale.

To prove (4.24) and (4.25), let $\phi \in \Phi$ arbitrary but fixed. From a basic estimate of the complex exponential function (proved in e.g. [26], Lemma 8.6, p.40) we have

$$\left| e^{if[\phi]} - 1 - if[\phi] \right| \leq \frac{|f[\phi]|^2}{2} \leq \frac{\rho(\phi)^2 \rho'(f)^2}{2} \leq \frac{\rho(\phi)^2}{2} < \infty, \quad \forall f \in B_{\rho'}(1).$$

Therefore, the functions $f \mapsto (e^{if[\phi]} - 1 - if[\phi])$ and $f \mapsto |f[\phi]|^2$ are bounded on $B_{\rho'}(1)$. Then, using similar arguments to those used to prove (4.30) we can show that

$$\lim_{A \in \mathcal{A}_{\rho'}} \int_A |f[\phi]|^2 \nu(df) = \int_{B_{\rho'}(1)} |f[\phi]|^2 \nu(df), \quad (4.31)$$

and

$$\lim_{A \in \mathcal{A}_{\rho'}} \int_A (e^{if[\phi]} - 1 - if[\phi]) \nu(df) = \int_{B_{\rho'}(1)} (e^{if[\phi]} - 1 - if[\phi]) \nu(df). \quad (4.32)$$

On the other hand, for any $A \in \mathcal{A}_{\rho'}$ and $T > 0$, we have that

$$\mathbb{E} \left(\sup_{t \in [0, T]} \left| M_t[\phi] - \tilde{J}_t(A)[\phi] \right|^2 \right) \leq q(\phi)^2 \mathbb{E} \left(\sup_{t \in [0, T]} q'(M_t - \tilde{J}_t(A))^2 \right). \quad (4.33)$$

Therefore, the fact that $\{\tilde{J}(A) : A \in \mathcal{A}_{\rho'}\}$ converges to M in $\mathcal{M}^2(\Phi'_q)$ and (4.33), implies that $\{\tilde{J}(A)[\phi] : A \in \mathcal{A}_{\rho'}\}$ converges to $M[\phi]$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ uniformly on compact intervals of $[0, \infty)$. This convergence together with (4.10) and (4.31) implies (4.25).

Furthermore, as for each $t \geq 0$, $\{\tilde{J}_t(A)[\phi] : A \in \mathcal{A}_{\rho'}\}$ converges to $M_t[\phi]$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then the net of characteristics functions $\{\mathbb{E}(\exp(i\tilde{J}_t(A)[\phi])) : A \in \mathcal{A}_{\rho'}\}$ converges to the characteristic function $\mathbb{E}(\exp(iM_t[\phi]))$ of M . Then, (4.9) and (4.32) implies (4.24).

Finally, as $\mathcal{M}^2(\Phi'_q)$ is metrizable, we can choose a subsequence $\{\tilde{J}^{A_n} : n \in \mathbb{N}\}$ that converges to M in $\mathcal{M}^2(\Phi'_q)$. Then, $\{\tilde{J}^{A_n} : n \in \mathbb{N}\}$ converges to M in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \Phi'_q)$ uniformly on compact intervals of $[0, \infty)$ and because each \tilde{J}^{A_n} is a Φ'_q -valued Lévy process, this implies that M is also a Φ'_q -valued Lévy process. This last fact implies that M is also a Φ'_β -valued Lévy process. \square

Notation 4.20. We denote by $\left\{ \int_{B_{\rho'}(1)} f \tilde{N}(t, df) : t \geq 0 \right\}$ the process $M = \{M_t\}_{t \geq 0}$ defined in Theorem 4.19.

The next result follows from Proposition 4.3 and because $\nu|_{B_{\rho'}(1)^c} \in \mathfrak{M}_R^b(\Phi'_\beta)$.

Proposition 4.21. *The Φ'_β -valued process $\left\{ \int_{B_{\rho'}(1)^c} f N(t, df) : t \geq 0 \right\}$ defined by*

$$\int_{B_{\rho'}(1)^c} f N(t, df)(\omega)[\phi] = \sum_{0 \leq s \leq t} \Delta L_s(\omega)[\phi] \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_s(\omega)), \quad \forall \omega \in \Omega, \phi \in \Phi. \quad (4.34)$$

is a $\{\mathcal{F}_t\}$ -adapted Φ'_β -valued regular càdlàg Lévy process. Moreover, $\forall \phi \in \Phi$,

$$\mathbb{E} \left(\exp \left\{ i \int_{B_{\rho'}(1)^c} f N(t, df)[\phi] \right\} \right) = \exp \left\{ t \int_{B_{\rho'}(1)^c} (e^{if[\phi]} - 1) \nu(df) \right\}. \quad (4.35)$$

Now, define the process $Y = \{Y_t\}_{t \geq 0}$ by

$$Y_t = L_t - \int_{B_{\rho'}(1)^c} f N(t, df), \quad \forall t \geq 0. \quad (4.36)$$

From Theorem 4.7 and Proposition 4.21 it follows that Y is a $\{\mathcal{F}_t\}$ -adapted Φ'_β -valued regular càdlàg Lévy process independent of $\left\{ \int_{B_{\rho'}(1)^c} f N(t, df) : t \geq 0 \right\}$. Moreover, from the definition of the Poisson integral (4.34), for any $0 \leq s < t$,

$$Y_t - Y_s = L_t - L_s - \sum_{s < u \leq t} \Delta L_u \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_u).$$

Therefore, $\sup_{t \geq 0} \rho'(\Delta Y_t(\omega)) \leq 1$ for each $\omega \in \Omega$. This in particular implies that for each $\phi \in \Phi$, the real-valued process $Y[\phi]$ satisfies, $\sup_{t \geq 0} |\Delta Y_t[\phi](\omega)| \leq \rho(\phi) < \infty$ for each $\omega \in \Omega$, thus $Y[\phi]$ has bounded jumps and consequently Y has finite moments to all orders (see [1], Theorem 2.4.7, p.118-9). Moreover, the independent and stationary increments of Y implies that for each $\phi \in \Phi$, the map $t \mapsto \mathbb{E}(Y_t[\phi])$ is additive and measurable. Therefore, there exists some $\mathbf{m} \in \Phi'_\beta$ such that $\mathbb{E}(Y_t[\phi]) = t\mathbf{m}[\phi]$, for all $\phi \in \Phi$, $t \geq 0$.

Now, consider the process $Z = \{Z_t\}_{t \geq 0}$ given by

$$Z_t = Y_t - t\mathbf{m}, \quad \forall t \geq 0. \quad (4.37)$$

From the properties of Y and the definition of \mathbf{m} , Z is a $\{\mathcal{F}_t\}$ -adapted Φ'_β -valued, zero-mean, càdlàg, regular Lévy process with moments to all orders and with jumps satisfying $\sup_{t \geq 0} \rho'(\Delta Z_t(\omega)) \leq 1$ for each $\omega \in \Omega$.

Now, for every $\phi \in \Phi$, let $\kappa(\phi) = \mathbb{E} \left[|Z_1[\phi]|^2 \right]$. The fact that Z_1 is a regular random variable with second moments shows that κ is a continuous Hilbertian semi-norm on Φ . Moreover, the independent and stationary increments of Z implies that $\mathbb{E} \left(|Z_t[\phi]|^2 \right) = t\kappa(\phi)^2$, for all $\phi \in \Phi$, $t \geq 0$. Hence, from Doob's inequality we have for every $T > 0$ that:

$$\mathbb{E} \left(\sup_{t \in [0, T]} |Z_t[\phi]|^2 \right) \leq 4\mathbb{E} \left(|Z_T[\phi]|^2 \right) = 4T\kappa(\phi)^2 \quad \forall \phi \in \Phi. \quad (4.38)$$

Theorem 4.22. *For the Φ'_β -valued process $X = \{X_t\}_{t \geq 0}$ defined by*

$$X_t = Z_t - \int_{B_{\rho'}(1)} f \tilde{N}(t, df), \quad \forall t \geq 0, \quad (4.39)$$

there exist a continuous Hilbertian semi-norm η on Φ and a Φ'_η -valued $\{\mathcal{F}_t\}$ -adapted Wiener process $W = \{W_t\}_{t \geq 0}$ with mean-zero and covariance functional \mathcal{Q} (as defined in Theorem 3.16) such that W is an indistinguishable version of X . Moreover, the semi-norm η can be chosen such that $\mathcal{Q} \leq K\eta$ (for some $K > 0$) and the map $i_{\mathcal{Q}, \eta}$ is Hilbert-Schmidt.

Proof. First, it is clear that X is a Φ'_β -valued $\{\mathcal{F}_t\}$ -adapted, càdlàg process that has zero-mean and square moments.

Now, we will show that for each $\phi \in \Phi$, the real-valued process $X[\phi] = \{X_t[\phi]\}_{t \geq 0}$ is a Wiener process. We proceed in a similar way as in the proof of Proposition 6.2 in [25], where a similar result for the separable Banach space case is considered.

First, let $\phi \in \Phi$ be such that $\rho(\phi) = 1$. As $Z[\phi]$ defines a real-valued càdlàg Lévy process it has a corresponding Lévy-Itô decomposition (see [1], Theorem 2.4.16, p.126) given by

$$Z_t[\phi] = b_\phi t + \sigma_\phi^2(B_\phi)_t + \int_{\{|y| \leq 1\}} y \tilde{N}_\phi(t, dy) + \int_{\{|y| > 1\}} y N_\phi(t, dy)$$

where $b_\phi \in \mathbb{R}$, $\sigma_\phi^2 \in \mathbb{R}_+$, B_ϕ is a standard real-valued Wiener process, N_ϕ is the Poisson random measure of $Z[\phi]$ and \tilde{N}_ϕ its compensated Poisson random measure. All the random components of the decomposition are independent. For a set $C \in \mathcal{B}(\mathbb{R})$ that is bounded below we have that

$$N_\phi(t, C)(\omega) = \sum_{0 \leq s \leq t} \mathbb{1}_C(\Delta Z_s(\omega)[\phi]) = \sum_{0 \leq s \leq t} \mathbb{1}_{\mathcal{Z}(\phi; C)}(\Delta Z_s(\omega)) = N_Z(t, \mathcal{Z}(\phi; C))(\omega),$$

where $\mathcal{Z}(\phi; C) := \{f \in \Phi' : f[\phi] \in C\}$, and N_Z denotes the Poisson random measure associated to Z . Note that $\mathcal{Z}(\phi; C)$ is a cylindrical set and consequently belongs to $\mathcal{B}(\Phi'_\beta)$. Moreover, as C is bounded below in $\mathcal{B}(\mathbb{R})$, it follows that $\mathcal{Z}(\phi; C)$ is bounded below in $\mathcal{B}(\Phi'_\beta)$. To see why this is true, let π_ϕ be given by (2.1). Then, by (2.2) and the continuity of π_ϕ it follows that $\overline{\mathcal{Z}(\phi; C)} = \overline{\pi_\phi^{-1}(C)} \subseteq \pi_\phi^{-1}(\overline{C})$. Hence, if $0 \in \overline{\mathcal{Z}(\phi; C)}$ then $0 \in \pi_\phi^{-1}(\overline{C})$, and consequently $0 \in \overline{C}$. But this contradicts the fact that C is bounded below. Therefore, $\mathcal{Z}(\phi; C)$ is bounded below.

Now, let $C = [-1, 1]^c$ and $D = \{f \in \Phi' : |f[\phi]| \leq 1\}$. We then have that $D = \mathcal{Z}(\phi; C)^c$ and because $\phi \in B_\rho(1)$, it follows that $B_{\rho'}(1) \subseteq D$. Now, because the jumps of Z satisfy $\sup_{t \geq 0} \rho'(\Delta Z_t(\omega)) \leq 1$ for each $\omega \in \Omega$, the support of $N_Z(t, \cdot)$ is in $B_{\rho'}(1)$ for each $t \geq 0$, and consequently the support of $\tilde{N}_Z(t, \cdot)$ is also in $B_\rho(1)$ for $t \geq 0$. Since $B_{\rho'}(1) \subseteq D$, it follows that

$$\int_D f \tilde{N}_Z(t, df)[\phi] = \int_{B_{\rho'}(1)} f \tilde{N}_Z(t, df)[\phi] + \int_{D \setminus B_{\rho'}(1)} f \tilde{N}_Z(t, df)[\phi] = \int_{B_{\rho'}(1)} f \tilde{N}_Z(t, df)[\phi]$$

and

$$\int_{D^c} f N_Z(t, df)[\phi] = 0.$$

Moreover, \tilde{N}_Z coincides with \tilde{N} in $B_{\rho'}(1)$, so we have that

$$\begin{aligned} Z_t[\phi] &= b_\phi t + \sigma_\phi^2 (B_\phi)_t + \int_{\{|y| < 1\}} y \tilde{N}_\phi(t, dy) + \int_{\{|y| \geq 1\}} y N_\phi(t, dy) \\ &= b_\phi t + \sigma_\phi^2 (B_\phi)_t + \int_D f \tilde{N}_Z(t, df)[\phi] + \int_{D^c} f N_Z(t, df)[\phi] \\ &= b_\phi t + \sigma_\phi^2 (B_\phi)_t + \int_{B_{\rho'}(1)} f \tilde{N}_Z(t, df)[\phi] \\ &= b_\phi t + \sigma_\phi^2 (B_\phi)_t + \int_{B_{\rho'}(1)} f \tilde{N}(t, df)[\phi] \end{aligned}$$

Now, taking expectations we obtain that for every $t \geq 0$,

$$0 = \mathbb{E} Z_t[\phi] = b_\phi t + \sigma_\phi^2 \mathbb{E}((B_\phi)_t) + \mathbb{E} \left(\int_{B_{\rho'}(1)} f \tilde{N}(t, df)[\phi] \right) = b_\phi t$$

consequently $b_\phi = 0$. We obtain $X_t[\phi] = Z_t[\phi] - \int_{B_{\rho'}(1)} f \tilde{N}(t, df)[\phi] = \sigma_\phi^2 (B_\phi)_t$ and so $X[\phi]$ is a Wiener process. The same representation holds for arbitrary $\phi \in \Phi$, as can be seen by replacing ϕ with $\phi/\rho(\phi)$ in the argument just given. Therefore, $X[\phi]$ is a Wiener process $\phi \in \Phi$.

Now, note that for every $T > 0$ and $\phi \in \Phi$, from Doob's inequality, (4.25) and (4.38), we have that

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T]} |X_t[\phi]|^2 \right) &\leq 4 \mathbb{E} \left(|X_T[\phi]|^2 \right) \\ &\leq 8T \left(\mathbb{E} \left(|Z_T[\phi]|^2 \right) + \mathbb{E} \left(|M_T[\phi]|^2 \right) \right) \\ &\leq 8T(\kappa(\phi)^2 + C_\rho q(\phi)^2), \end{aligned}$$

where $C_\rho = \int_{B_{\rho'}(1)} q'(f)^2 \nu(df) < \infty$. Let σ be a continuous Hilbertian semi-norm on Φ such that $\kappa \leq \sigma$ and $q \leq \sigma$. Then, from the above inequalities for each $T > 0$ and $\phi \in \Phi$ we have

$$\mathbb{E} \left(\sup_{t \in [0, T]} |X_t[\phi]|^2 \right) \leq 8T(1 + C_\rho)\sigma(\phi)^2.$$

Then, Theorem 3.9 shows that there exists a continuous Hilbertian semi-norm η on Φ , $\sigma \leq \eta$, such that $i_{\sigma, \eta}$ is Hilbert-Schmidt and there exists a Φ'_η -valued Wiener processes (i.e. a continuous Lévy process) $W = \{W_t\}_{t \geq 0}$ that has finite second moments in Φ'_η and such that for every $\phi \in \Phi$, $W[\phi] = \{W_t[\phi]\}_{t \geq 0}$ is a version of $X[\phi] = \{X_t[\phi]\}_{t \geq 0}$. However, as both W and X are regular càdlàg processes in Φ'_β , then the fact that $W[\phi] = X[\phi]$ for each $\phi \in \Phi$ implies that W and X are indistinguishable (Proposition 2.3). Hence, W is $\{\mathcal{F}_t\}$ -adapted and is also Φ'_β -valued Wiener process.

Finally, if \mathcal{Q} is the covariance functional of W , from (3.19) it follows that for every $\phi \in \Phi$ we have

$$\mathcal{Q}(\phi)^2 = \mathbb{E}(|W_1[\phi]|^2) = \mathbb{E}(|X_1[\phi]|^2) \leq 2(1 + C_\rho)\sigma(\phi)^2 \leq 2(1 + C_\rho)\eta(\phi)^2.$$

Then, $\mathcal{Q} \leq K\eta$ with $K^2 = 2(1 + C_\rho)$. Moreover, because $i_{\mathcal{Q}, \sigma}$ is linear and continuous and $i_{\sigma, \eta}$ is Hilbert-Schmidt, we have that $i_{\mathcal{Q}, \eta} = i_{\sigma, \eta} \circ i_{\mathcal{Q}, \sigma}$ is Hilbert-Schmidt. \square

We are ready for the main result of this section.

Theorem 4.23 (Lévy-Itô decomposition). *Let $L = \{L_t\}_{t \geq 0}$ be a Φ'_β -valued Lévy process. Then, for each $t \geq 0$ it has the following representation*

$$L_t = t\mathbf{m} + W_t + \int_{B_{\rho'}(1)} f \tilde{N}(t, df) + \int_{B_{\rho'}(1)^c} f N(t, df) \quad (4.40)$$

where

- (1) $\mathbf{m} \in \Phi'_\beta$,
- (2) ρ is a continuous Hilbertian semi-norm on Φ such that the Lévy measure ν of L satisfies (4.23) and $B_{\rho'}(1) := \{f \in \Phi'_\beta : \rho'(f) \leq 1\}$ is a bounded, closed, convex, balanced subset of Φ'_β ,
- (3) $\{W_t\}_{t \geq 0}$ is a Φ'_η -valued Wiener process with mean-zero and covariance functional \mathcal{Q} , where η is a continuous Hilbertian semi-norm on Φ such that $\mathcal{Q} \leq K\eta$ (for some $K > 0$) and the map $i_{\mathcal{Q}, \eta}$ is Hilbert-Schmidt,
- (4) $\left\{ \int_{B_{\rho'}(1)} f \tilde{N}(t, df) : t \geq 0 \right\}$ is a Φ'_q -valued mean-zero, square integrable, càdlàg Lévy process with characteristic function given by (4.24) and second moments given by (4.25), where q is a continuous Hilbertian semi-norm on Φ such that $\rho \leq q$ and the map $i_{\rho, q}$ is Hilbert-Schmidt,
- (5) $\left\{ \int_{B_{\rho'}(1)^c} f N(t, df) : t \geq 0 \right\}$ is a Φ'_β -valued càdlàg Lévy process with characteristic function given by (4.35).

All the random components of the decomposition (4.40) are independent.

Proof. The decomposition (4.40) and the properties of its components follows from Theorems 4.19 and 4.22, Proposition 4.21, (4.36) and (4.37). Now we prove the independence of the components in (4.40).

For any $\phi_1, \dots, \phi_n \in \Phi$, by considering the Lévy-Itô decomposition of the \mathbb{R}^n -valued Lévy process $\{(L_t[\phi_1], \dots, L_t[\phi_n])\}_{t \geq 0}$, it follows that the \mathbb{R}^n -valued processes

$$\{(W_t[\phi_1], \dots, W_t[\phi_n])\}_{t \geq 0}, \left\{ \left(\int_{B_{\rho'}(1)} f \tilde{N}(t, df)[\phi_1], \dots, \int_{B_{\rho'}(1)} f \tilde{N}(t, df)[\phi_n] \right) : t \geq 0 \right\},$$

and $\left\{ \left(\int_{B_{\rho'}(1)^c} f N(t, df)[\phi_1], \dots, \int_{B_{\rho'}(1)^c} f N(t, df)[\phi_n] \right) : t \geq 0 \right\}$ are independent. But because the processes $\{W_t\}_{t \geq 0}$, $\left\{ \int_{B_{\rho'}(1)} f \tilde{N}(t, df) : t \geq 0 \right\}$ and $\left\{ \int_{B_{\rho'}(1)} f N(t, df) : t \geq 0 \right\}$ are regular, then Proposition 2.4 shows that they are independent. \square

As an important by-product of the proof of the Lévy-Itô decomposition we obtain a Lévy-Khintchine theorem for the characteristic function of any Φ'_β -valued Lévy process.

Theorem 4.24 (Lévy-Khintchine theorem for Φ'_β -valued Lévy processes).

(1) If $L = \{L_t\}_{t \geq 0}$ is a Φ'_β -valued, regular, càdlàg Lévy process, there exist $\mathbf{m} \in \Phi'_\beta$, a continuous Hilbertian semi-norm \mathcal{Q} on Φ , a Lévy measure ν on Φ'_β and a continuous Hilbertian semi-norm ρ on Φ for which ν satisfies (4.23); and such that for each $t \geq 0$, $\phi \in \Phi$,

$$\begin{aligned} \mathbb{E} \left(e^{iL_t[\phi]} \right) &= e^{t\eta(\phi)}, \quad \text{with} \\ \eta(\phi) &= i\mathbf{m}[\phi] - \frac{1}{2}\mathcal{Q}(\phi)^2 + \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1 - if[\phi] \mathbb{1}_{B_{\rho'}(1)}(f) \right) \nu(df). \end{aligned} \quad (4.41)$$

(2) Conversely, let $\mathbf{m} \in \Phi'_\beta$, \mathcal{Q} be a continuous Hilbertian semi-norm on Φ , and ν be a θ -regular Lévy measure on Φ'_β satisfying (4.23) for a continuous Hilbertian semi-norm ρ on Φ . There exists a Φ'_β -valued, regular, càdlàg Lévy process $L = \{L_t\}_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, unique up to equivalence in distribution, whose characteristic function is given by (4.41). In particular, ν is the Lévy measure of L .

Proof. If L is a Φ'_β -valued, regular, càdlàg Lévy process then (4.41) follows from the independence of the random components of the decomposition (4.40), (3.20) (recall here that W has mean zero and covariance functional \mathcal{Q}), (4.24) and (4.35).

For the converse, assume we have \mathbf{m} , \mathcal{Q} , ν and ρ with the properties in the statement of the theorem. First, as ν is a σ -finite Borel measure on Φ'_β (Proposition 4.10), there exist a stationary Poisson point processes $p = \{p(t)\}_{t \geq 0}$ on $(\Phi'_\beta, \mathcal{B}(\Phi'_\beta))$ with associated Poisson random measure R , p and R unique up to equivalence in distribution, such that ν is the characteristic measure of p (see [14], Theorem I.9.1, p.44. See also [26], Proposition 19.4, p.122). If $U_n \in \mathcal{B}(\Phi'_\beta)$, for $n \in \mathbb{N}$, are disjoint, $\Phi'_\beta = \bigcup_n U_n$ and $\nu(U_n) < \infty$ for every $n \in \mathbb{N}$, the point process p can be constructed from a sequence of stopping times $\tau_i^{(n)}$ with exponential distribution with parameter $\nu(U_n)$ and a sequence $\xi_i^{(n)}$ of Φ'_β -valued random variables with probability distribution $\nu(\cdot)/\nu(U_n)$ (see details in [14], Theorem I.9.1, p.44). Because ν is concentrated on Φ'_θ for a weaker countably Hilbertian topology θ on Φ (Lemma 4.6), it follows that the random variables $\xi_i^{(n)}$ are regular. But as p takes the values of these random variables (indeed we have $p(\tau_1^{(n)} + \cdot + \tau_i^{(n)}) = \xi_i^{(n)}$ for $n, i \in \mathbb{N}$), then p is a regular process in Φ'_β .

Now, note that in the proof of Theorem 4.19, we only used the fact that the Lévy measure ν of a Lévy process L satisfies the integrability condition in (4.23), and that the Poisson integral with respect to the Poisson random measure N of L exists and satisfies the properties given in Section 4.1. Since we can define Poisson integrals with respect to the Poisson measure R of p satisfying the properties given in Section 4.1 (here we use that p is a regular process), and ν satisfies (4.23), we can replicate the arguments in the proof of Theorem 4.19 to conclude that there exists a continuous Hilbertian semi-norm q on Φ such that $\rho \leq q$ and the map $i_{\rho, q}$ is Hilbert-Schmidt, and a Φ'_q -valued mean-zero, square integrable, càdlàg Lévy process $\widetilde{M} = \{\widetilde{M}_t\}_{t \geq 0}$ with characteristic function given by (4.24).

On the other hand, because from (4.23) we have $\nu(B_{\rho'}(1)^c) < \infty$, it follows from Proposition 4.21 that there exists a Φ'_β -valued, regular, càdlàg Lévy process $\widetilde{J} = \{\widetilde{J}_t\}_{t \geq 0}$, where $\widetilde{J}_t = \int_{B_{\rho'}(1)^c} fR(t, df)$ as given in (4.34) (with N replaced by R), with characteristic function (4.35). Moreover, from Theorem 3.17 there exists a Φ'_β -valued Wiener process $\widetilde{W} = \{\widetilde{W}_t\}_{t \geq 0}$, unique up to equivalence in distribution, such that \mathbf{m} and \mathcal{Q} are the mean and the covariance functional of \widetilde{W} . Hence, \widetilde{W} has characteristic function given by (3.20).

We can assume without loss of generality that \widetilde{W} , \widetilde{M} and \widetilde{J} are independent Φ'_β -valued process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see e.g. [16], Corollary 6.18, p.117). Hence, if we define $L = \{L_t\}_{t \geq 0}$, where for each $t \geq 0$, $L_t = \widetilde{W}_t + \widetilde{M}_t + \widetilde{J}_t$, then L being the sum of a finite number of independent càdlàg Lévy process is also a Φ'_β -valued, càdlàg Lévy process. It is also unique up to equivalence in distribution, and for each $t \geq 0$, L_t has characteristic function given by (4.41). \square

In view of Theorem 4.24(2), if L is a Φ'_β -valued Lévy process with characteristic function (4.41), then the members of the array $(\mathbf{m}, \mathcal{Q}, \nu, \rho)$, called the *characteristics* of L , determine uniquely (up to equivalence in distribution) the Lévy process L .

5 Lévy-Khintchine theorem for infinitely divisible measures

Theorem 5.1 (Lévy-Khintchine theorem). *Let $\mu \in \mathfrak{M}_R^1(\Phi'_\beta)$. Then:*

- (1) *If Φ is also a barrelled space and if μ is infinitely divisible, then there exists $\mathbf{m} \in \Phi'_\beta$, a continuous Hilbertian semi-norm \mathcal{Q} on Φ , a Lévy measure ν on Φ'_β and a continuous Hilbertian semi-norm ρ on Φ for which ν satisfies (4.23); such that the characteristic function of μ satisfies the following formula for every $\phi \in \Phi$:*

$$\widehat{\mu}(\phi) = \exp \left[i\mathbf{m}[\phi] - \frac{1}{2}\mathcal{Q}(\phi)^2 + \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1 - if[\phi]\mathbb{1}_{B_{\rho'}(1)}(f) \right) \nu(df) \right]. \quad (5.1)$$

- (2) *Conversely, let $\mathbf{m} \in \Phi'_\beta$, \mathcal{Q} be a continuous Hilbertian semi-norm on Φ , and ν be a θ -regular Lévy measure on Φ'_β satisfying (4.23) for a continuous Hilbertian semi-norm ρ on Φ . If μ has characteristic function given by (5.1), then μ is infinitely divisible.*

Proof. First, suppose that μ is infinitely divisible. Then, it follows from Theorem 3.14 that there exists a Φ'_β -valued, regular, càdlàg Lévy process $L = \{L_t\}_{t \geq 0}$ such that $\mu_{L_1} = \mu$. Then, the existence of μ , \mathcal{Q} , ν and ρ follows from Theorem 4.24(1). Furthermore, the fact that μ satisfies (5.1) follows from taking $t = 1$ in (4.41) and because $\mu_{L_1} = \mu$.

Conversely, suppose that μ satisfies (5.1) for the given μ , \mathcal{Q} , ν and ρ . Then it follows from Theorem 4.24(2) that there exists a Φ'_β -valued, regular, càdlàg Lévy process $L = \{L_t\}_{t \geq 0}$ such that $\mu_{L_1} = \mu$. But then Theorem 3.5 shows that μ is infinitely divisible. \square

Remark 5.2. *If Φ is a barrelled nuclear space, the assumption that the Lévy measure ν is θ -regular in Theorems 4.24(2) and 5.1(2) can be disposed because every Lévy measure on Φ is θ -regular (see Corollary 4.12).*

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